

# Local zeta regularization and the scalar Casimir effect II. Some explicitly solvable cases

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## Abstract

In Part I of this series of papers we have described a general formalism to compute the vacuum effects of a scalar field via local (or global) zeta regularization. In the present Part II we exemplify the general formalism in a number of cases which can be solved explicitly by analytical means. More in detail we deal with configurations involving parallel or perpendicular planes and we also discuss the case of a three-dimensional wedge.

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# 1 Introduction

This is the second part of our series of papers about zeta regularization and vacuum effects for a scalar field. In Part I [19] we have considered a neutral, scalar quantum field on  $(d+1)$ -dimensional Minkowski spacetime, assuming the field to be confined within a spatial domain  $\Omega$  and to fulfill suitable boundary conditions; we have also indicated the possibility to replace  $\Omega$  with a Riemannian manifold, or either an open subset of it (and the Minkowski environment with a curved ultrastatic spacetime). In the same work we have been presenting general methods for the zeta regularization of the vacuum expectation value (VEV) of the stress-energy tensor, of the total energy and of the boundary forces, emphasizing the fact that we were producing a set of general rules to be applied almost mechanically in specific configurations. The illustration of such mechanical rules was started in Part I with some simple examples in one spatial dimension, and it is continued in the present Part II with more engaging configurations. Here we consider a number of cases, in which the necessary computations can be carried over by purely analytical means; this makes a difference with respect to the subsequent Parts III [20] and IV [21], where application of the general rules will require a mixture of analytical and numerical methods.

Most of the completely solvable cases considered here have been previously treated in the literature, typically in the case of spatial dimension three, often with special choices of the boundary conditions and, in most cases, considering only the conformal part of the stress-energy VEV; for each one of the cases already treated a specific approach has been employed, possibly different from zeta regularization.

The present work attains more generality as for the spatial dimension, the boundary conditions and the presence of a non-conformal part in the stress-energy VEV; the second, perhaps more significant, contribution of this paper is the unified view-point mentioned in the previous comments, that is, the presentation of all cases as applications of the same apparatus.

To be more specific, the configurations that we analyse are the following.

i) First of all we consider a massless field in odd spatial dimension confined between two parallel (hyper-)planes, for several kinds of boundary conditions; see Section 3. We have already considered this model in spatial dimension  $d = 3$  for Dirichlet boundary conditions in [17]; therein, the analytic continuation required by zeta regularization was performed using ad hoc, known results on the special functions related to this specific configuration (namely, the Riemann zeta and the polylogarithm). As a matter of fact, the literature on the configuration with two parallel planes is immense, both regarding local and global aspects; here we only cite a few references. In his seminal paper [9], using an exponential cut-off regularization along with Abel-Plana resummation, Casimir was the first to compute the total energy and the boundary forces for the case of two parallel planes; concerning local aspects,

the foremost derivation of the full stress-energy tensor VEV was given by Brown and Maclay [8], using a point-splitting technique <sup>(2)</sup>. Computation of both global and local quantities for this model was later reproposed by several authors, using various regularization techniques: see, e.g., the monographies by Milton [25], Elizalde et al. [14, 15], Bordag et al. [4] (see, as well, the works cited therein) and the papers by Zimmerman et al. [30] and by Esposito et al. [16]. We will give further bibliographical references in Section 3. In the present paper we resort to the formalism of integral kernels developed in Part I in order to derive automatically the analytic continuations required by the use of zeta regularization.

ii) Next, we consider a massive field between an arbitrary number of perpendicular (hyper-)planes, fulfilling either Dirichlet or Neumann boundary conditions on each one of them; see Section 4. To the best of our knowledge, this type of configuration was only considered by Actor [1] and by Actor and Bender [2]; both these papers deal with a scalar field fulfilling Dirichlet boundary conditions on at most three perpendicular planes (along with other similar models in three spatial dimensions, all involving boundaries consisting of flat, perpendicular parts). More precisely, in [1] the author fixes  $d = 3$  and evaluates the renormalized effective Lagrangian, along with the VEV  $\langle 0|\hat{\phi}^2(x)|0\rangle_{ren}$  (plus their analogues for non-zero temperature); in [2]  $d$  is arbitrary and it is computed the VEV of each component of the stress-energy tensor. In both works cited above, a zeta type approach based on the use of heat kernels is employed; however, contrary to our regularization scheme, this approach also involves the subtraction of terms (corresponding essentially to Minkowski spacetime contributions) which diverge for any value of the regulator parameter. Besides, let us stress once more that our method is more systematic and the construction of the required analytic continuations descends automatically from a general framework.

iii) Finally, we consider a massless field confined within a wedge of arbitrary width in spatial dimension  $d = 3$ , for several types of boundary conditions; see Section 5. We also consider a variation of this configuration, which corresponds essentially to identify the sides of the wedge; this is the so-called case of the “cosmic string” (see subsection 5.9). Some of these cases have already been treated by Dowker et al. [12, 13], Deutsch and Candelas [11] (also discussing the electromagnetic case) and, more recently, by Saharian et al. [29, 31] and by Fulling et al. [23] (see also the citations in these works and [6, 7, 26]); nearly all of these authors use the point splitting approach, or some variant of it. More in detail, in [12] and [11] attention is restricted to the conformal part of the stress-energy VEV for either Dirichlet or Neumann boundary conditions, while in [29, 31] also the non-conformal part is considered, but in the Dirichlet case only; in [23], instead, the authors only show the graphs of the energy density and of the pressure components (for which no explicit

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<sup>2</sup>Actually, both Casimir and Brown-Maclay considered the case of electromagnetic field; yet the methods employed by these authors can be trivially adapted to the case of a scalar field.

expression is given), derived via a point-splitting approach for several configurations and various choices of the parameters describing the theory. Our approach via zeta regularization considers several types of boundary conditions, both in the conformal and in the non-conformal case; in the subcases already analysed in the literature cited above, we obtain the same results.

Before describing the applications (i-iii), in the forthcoming Section 2 we summarize the general scheme of Part I (more precisely, the parts of it required by the applications analysed in the present paper). This summary has been written just for the comfort of the reader who, in absence of it, would be forced to skip continuously to Part I to recover the basic identities on integral kernels and analytic continuation employed here.

Finally, we point out that some of the computations presented in this paper have been performed using `Mathematica` in the symbolic mode.

## 2 A summary of results from Part I

**2.1 General setting.** Throughout the paper we use natural units, so that

$$c = 1 \ , \quad \hbar = 1 \ . \quad (2.1)$$

Our approach works in  $(d+1)$ -dimensional Minkowski spacetime, which is identified with  $\mathbf{R}^{d+1}$  using a set of inertial coordinates

$$x = (x^\mu)_{\mu=0,1,\dots,d} \equiv (x^0, \mathbf{x}) \equiv (t, \mathbf{x}) \ ; \quad (2.2)$$

the Minkowski metric is  $(\eta_{\mu\nu}) = \text{diag}(-1, 1, \dots, 1)$ . We fix a spatial domain  $\Omega \subset \mathbf{R}^d$  and a background static potential  $V: \Omega \rightarrow \mathbf{R}$ . We consider a quantized neutral, scalar field  $\hat{\phi}: \mathbf{R} \times \Omega \rightarrow \mathcal{L}_{sa}(\mathfrak{F})$  ( $\mathfrak{F}$  is the Fock space and  $\mathcal{L}_{sa}(\mathfrak{F})$  are the selfadjoint operators on it); suitable boundary conditions are prescribed on  $\partial\Omega$ . The field equation reads

$$0 = (-\partial_{tt} + \Delta - V(\mathbf{x}))\hat{\phi}(\mathbf{x}, t) \quad (2.3)$$

( $\Delta := \sum_{i=1}^d \partial_{ii}$  is the  $d$ -dimensional Laplacian). We put

$$\mathcal{A} := -\Delta + V \ , \quad (2.4)$$

keeping into account the boundary conditions on  $\partial\Omega$ , and consider the Hilbert space  $L^2(\Omega)$  with inner product  $\langle f|g \rangle := \int_\Omega d\mathbf{x} \, \overline{f}(\mathbf{x})g(\mathbf{x})$ . We assume  $\mathcal{A}$  to be selfadjoint in  $L^2(\Omega)$  and strictly positive (i.e., with spectrum  $\sigma(\mathcal{A}) \subset [\varepsilon^2, +\infty)$  for some  $\varepsilon > 0$ ); the latter assumption is sometimes relaxed requiring only that  $\mathcal{A}$  is non-negative ( $\sigma(\mathcal{A}) \subset [0, +\infty)$ ).

We often refer to a complete orthonormal set  $(F_k)_{k \in \mathcal{K}}$  of (proper or improper) eigenfunctions of  $\mathcal{A}$  with eigenvalues  $(\omega_k^2)_{k \in \mathcal{K}}$  ( $\omega_k \geq \varepsilon$  for all  $k \in \mathcal{K}$ ; with the relaxed condition of non-negativity, we only have  $\omega_k \geq 0$ ). Thus

$$\begin{aligned} F_k : \Omega &\rightarrow \mathbf{C}; & \mathcal{A}F_k &= \omega_k^2 F_k; \\ \langle F_k | F_h \rangle &= \delta(k, h) & \text{for all } k, h \in \mathcal{K}. \end{aligned} \quad (2.5)$$

The labels  $k \in \mathcal{K}$  can include both discrete and continuous parameters;  $\int_{\mathcal{K}} dk$  indicates summation over all labels and  $\delta(h, k)$  is the Dirac delta function on  $\mathcal{K}$ .

We expand the field  $\hat{\phi}$  in terms of destruction and creation operators corresponding to the above eigenfunctions, and assume the canonical commutation relations;  $|0\rangle \in \mathfrak{F}$  is the vacuum state and, as already indicated, VEV stands for “vacuum expectation value”.

The quantized stress-energy tensor reads ( $\xi \in \mathbf{R}$  is a parameter)

$$\hat{T}_{\mu\nu} := (1 - 2\xi) \partial_\mu \hat{\phi} \circ \partial_\nu \hat{\phi} - \left( \frac{1}{2} - 2\xi \right) \eta_{\mu\nu} (\partial^\lambda \hat{\phi} \partial_\lambda \hat{\phi} + V \hat{\phi}^2) - 2\xi \hat{\phi} \circ \partial_{\mu\nu} \hat{\phi}; \quad (2.6)$$

in the above we put  $\hat{A} \circ \hat{B} := (1/2)(\hat{A}\hat{B} + \hat{B}\hat{A})$  for all  $\hat{A}, \hat{B} \in \mathcal{L}_{sa}(\mathfrak{F})$ , and all the bilinear terms in the field are evaluated on the diagonal (e.g.,  $\partial_\mu \hat{\phi} \circ \partial_\nu \hat{\phi}$  indicates the map  $x \mapsto \partial_\mu \hat{\phi}(x) \circ \partial_\nu \hat{\phi}(x)$ ). The VEV  $\langle 0 | \hat{T}_{\mu\nu} | 0 \rangle$  is typically divergent.

**2.2 Zeta regularization.** The *zeta-regularized field operator* is

$$\hat{\phi}^u := (\kappa^{-2} \mathcal{A})^{-u/4} \hat{\phi}, \quad (2.7)$$

where  $\mathcal{A}$  is the operator (2.4),  $u \in \mathbf{C}$  and  $\kappa > 0$  is a “mass scale” parameter; note that  $\hat{\phi}^u|_{u=0} = \hat{\phi}$ , at least formally. The *zeta regularized stress-energy tensor* is

$$\hat{T}_{\mu\nu}^u := (1 - 2\xi) \partial_\mu \hat{\phi}^u \circ \partial_\nu \hat{\phi}^u - \left( \frac{1}{2} - 2\xi \right) \eta_{\mu\nu} (\partial^\lambda \hat{\phi}^u \partial_\lambda \hat{\phi}^u + V(\hat{\phi}^u)^2) - 2\xi \hat{\phi}^u \circ \partial_{\mu\nu} \hat{\phi}^u. \quad (2.8)$$

The VEV  $\langle 0 | \hat{T}_{\mu\nu}^u | 0 \rangle$  is well defined for  $\Re u$  large enough (see the forthcoming subsection 2.5); moreover, in the region of definition it is an analytic function of  $u$ . The same can be said of many related observables (including global objects, such as the total energy VEV).

For any one of these observables, let us denote with  $\mathcal{F}(u)$  its zeta-regularized version and assume this to be analytic for  $u$  in a suitable domain  $\mathcal{U}_0 \subset \mathbf{C}$ . The zeta approach to renormalization can be formulated in two versions.

i) *Restricted version.* Assume the map  $\mathcal{U}_0 \rightarrow \mathbf{C}$ ,  $u \mapsto \mathcal{F}(u)$  to admit an analytic continuation (indicated with the same notation) to an open subset  $\mathcal{U} \subset \mathbf{C}$  with  $\mathcal{U} \ni 0$ ; then we define the renormalized observables as

$$\mathcal{F}_{ren} := \mathcal{F}(0). \quad (2.9)$$

ii) *Extended version.* Assume that there exists an open subset  $\mathcal{U} \subset \mathbf{C}$  with  $\mathcal{U}_0 \subset \mathcal{U}$ , such that  $0 \in \mathcal{U}$  and the map  $u \in \mathcal{U}_0 \mapsto \mathcal{F}(u)$  has an analytic continuation to  $\mathcal{U} \setminus \{0\}$  (still denoted with  $\mathcal{F}$ ). Starting from the Laurent expansion  $\mathcal{F}(u) = \sum_{k=-\infty}^{+\infty} \mathcal{F}_k u^k$ , we introduce the *regular part*  $(RP \mathcal{F})(u) := \sum_{k=0}^{+\infty} \mathcal{F}_k u^k$  and define

$$\mathcal{F}_{ren} := (RP \mathcal{F})(0) . \quad (2.10)$$

Of course, if  $\mathcal{F}$  is regular at  $u = 0$  the definitions (2.9) (2.10) coincide. In the case of the stress-energy VEV, the prescriptions (i) and (ii) read, respectively,

$$\langle 0 | \widehat{T}_{\mu\nu}(x) | 0 \rangle_{ren} := \langle 0 | \widehat{T}_{\mu\nu}^u(x) | 0 \rangle \Big|_{u=0} , \quad (2.11)$$

$$\langle 0 | \widehat{T}_{\mu\nu}(x) | 0 \rangle_{ren} := RP \Big|_{u=0} \langle 0 | \widehat{T}_{\mu\nu}^u(x) | 0 \rangle . \quad (2.12)$$

**2.3 Conformal and non-conformal parts of the stress-energy VEV.** These are indicated by the superscripts  $(\diamond)$  and  $(\blacksquare)$ , respectively; they are defined by

$$\langle 0 | \widehat{T}_{\mu\nu} | 0 \rangle_{ren} = \langle 0 | \widehat{T}_{\mu\nu}^{(\diamond)} | 0 \rangle_{ren} + (\xi - \xi_d) \langle 0 | \widehat{T}_{\mu\nu}^{(\blacksquare)} | 0 \rangle_{ren} , \quad (2.13)$$

where we are considering for the parameter  $\xi$  the critical value

$$\xi_d := \frac{d-1}{4d} . \quad (2.14)$$

**2.4 Integral kernels.** If  $\mathcal{B}$  is a linear operator in  $L^2(\Omega)$ , its integral kernel is the (generalized) function  $(\mathbf{x}, \mathbf{y}) \in \Omega \times \Omega \mapsto \mathcal{B}(\mathbf{x}, \mathbf{y}) := \langle \delta_{\mathbf{x}} | \mathcal{B} \delta_{\mathbf{y}} \rangle$  ( $\delta_{\mathbf{x}}$  is the Dirac delta at  $\mathbf{x}$ ). The trace of  $\mathcal{B}$ , assuming it exists, fulfills  $\text{Tr } \mathcal{B} = \int_{\Omega} d\mathbf{x} \mathcal{B}(\mathbf{x}, \mathbf{x})$ .

In the following subsections  $\mathcal{A}$  is a strictly positive selfadjoint operator in  $L^2(\Omega)$ , with a complete orthonormal set of eigenfunctions as in Eq. (2.5); in some situations (explicitly indicated) we only require  $\mathcal{A}$  to be non-negative. In typical applications,  $\mathcal{A}$  is the operator (2.4).

**2.5 The Dirichlet kernel and its relations with the stress-energy VEV.** For (suitable)  $s \in \mathbf{C}$ , the  $s$ -th Dirichlet kernel of  $\mathcal{A}$  is

$$D_s(\mathbf{x}, \mathbf{y}) := \mathcal{A}^{-s}(\mathbf{x}, \mathbf{y}) = \int_{\mathcal{K}} \frac{dk}{\omega_k^{2s}} F_k(\mathbf{x}) \overline{F_k}(\mathbf{y}) . \quad (2.15)$$

If  $\mathcal{A} = -\Delta + V$  (with  $V$  a smooth potential) is strictly positive, the map  $D_s(, ) : \Omega \times \Omega \rightarrow \mathbf{C}$ ,  $(\mathbf{x}, \mathbf{y}) \mapsto D_s(\mathbf{x}, \mathbf{y})$  is continuous along with all its partial derivatives up to order  $j \in \mathbf{N}$ , for all  $s \in \mathbf{C}$  with  $\Re s > d/2 + j/2$ ; in particular,  $D_s$  and its derivatives up to order  $j$  are continuous on the diagonal  $\mathbf{y} = \mathbf{x}$ , a fact of special interest for our purposes. (Under stronger assumptions on  $\mathcal{A}$ , one can give results



of absolute and uniform convergence of the eigenfunction expansion in Eq. (2.15), for suitable values of  $s$ ; the same can be said for all the corresponding derivatives). Recalling Eq. (2.8), the regularized stress-energy VEV can be expressed as follows:

$$\langle 0|\widehat{T}_{00}^u(\mathbf{x})|0\rangle = \kappa^u \left[ \left( \frac{1}{4} + \xi \right) D_{\frac{u-1}{2}}(\mathbf{x}, \mathbf{y}) + \left( \frac{1}{4} - \xi \right) (\partial^{x^\ell} \partial_{y^\ell} + V(\mathbf{x})) D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \right]_{\mathbf{y}=\mathbf{x}}, \quad (2.16)$$

$$\langle 0|\widehat{T}_{0j}^u(\mathbf{x})|0\rangle = \langle 0|\widehat{T}_{j0}^u(\mathbf{x})|0\rangle = 0, \quad (2.17)$$

$$\begin{aligned} \langle 0|\widehat{T}_{ij}^u(\mathbf{x})|0\rangle &= \langle 0|\widehat{T}_{ji}^u(\mathbf{x})|0\rangle = \\ &= \kappa^u \left[ \left( \frac{1}{4} - \xi \right) \delta_{ij} \left( D_{\frac{u-1}{2}}(\mathbf{x}, \mathbf{y}) - (\partial^{x^\ell} \partial_{y^\ell} + V(\mathbf{x})) D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \right) + \right. \\ &\quad \left. + \left( \left( \frac{1}{2} - \xi \right) \partial_{x^i y^j} - \xi \partial_{x^i x^j} \right) D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \right]_{\mathbf{y}=\mathbf{x}} \end{aligned} \quad (2.18)$$

( $\langle 0|\widehat{T}_{\mu\nu}^u(\mathbf{x})|0\rangle$  is short for  $\langle 0|\widehat{T}_{\mu\nu}^u(t, \mathbf{x})|0\rangle$ ; indeed, the VEV does not depend on  $t$ ). Of course, the map  $\Omega \rightarrow \mathbf{C}$ ,  $\mathbf{x} \mapsto \langle 0|\widehat{T}_{\mu\nu}^u(\mathbf{x})|0\rangle$  possesses the same regularity as the functions  $\mathbf{x} \in \Omega \mapsto D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{x})$ ,  $\partial_{zw} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{x})$  ( $z, w$  any two spatial variables); so, due to the previously mentioned results,  $\mathbf{x} \mapsto \langle 0|\widehat{T}_{\mu\nu}^u(\mathbf{x})|0\rangle$  is continuous for  $\Re u > d + 1$ .

The renormalized stress-energy VEV is  $\langle 0|\widehat{T}_{\mu\nu}^u(\mathbf{x})|0\rangle_{ren} := RP|_{u=0} \langle 0|\widehat{T}_{\mu\nu}^u(\mathbf{x})|0\rangle$ ; introducing the functions

$$D_{\pm\frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) := RP|_{u=0} \left( \kappa^u D_{\frac{u\pm 1}{2}}(\mathbf{x}, \mathbf{y}) \right), \quad (2.19)$$

$$\partial_{zw} D_{\frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) := RP|_{u=0} \left( \kappa^u \partial_{zw} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \right) \quad (2.20)$$

(with  $z, w$  any two spatial variables), this can be expressed as follows:

$$\langle 0|\widehat{T}_{00}^u(\mathbf{x})|0\rangle_{ren} = \left[ \left( \frac{1}{4} + \xi \right) D_{-\frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) + \left( \frac{1}{4} - \xi \right) (\partial^{x^\ell} \partial_{y^\ell} + V(\mathbf{x})) D_{+\frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) \right]_{\mathbf{y}=\mathbf{x}}, \quad (2.21)$$

$$\langle 0|\widehat{T}_{0j}^u(\mathbf{x})|0\rangle_{ren} = \langle 0|\widehat{T}_{j0}^u(\mathbf{x})|0\rangle_{ren} = 0, \quad (2.22)$$

$$\begin{aligned} \langle 0|\widehat{T}_{ij}^u(\mathbf{x})|0\rangle_{ren} &= \langle 0|\widehat{T}_{ji}^u(\mathbf{x})|0\rangle_{ren} = \\ &= \left[ \left( \frac{1}{4} - \xi \right) \delta_{ij} \left( D_{-\frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) - (\partial^{x^\ell} \partial_{y^\ell} + V(\mathbf{x})) D_{+\frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) \right) + \right. \\ &\quad \left. + \left( \left( \frac{1}{2} - \xi \right) \partial_{x^i y^j} - \xi \partial_{x^i x^j} \right) D_{+\frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) \right]_{\mathbf{y}=\mathbf{x}}. \end{aligned} \quad (2.23)$$

If  $D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y})$  and  $\partial_{zw} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y})$  have analytic continuations regular at  $u = 0$ , indicated with  $D_{\pm\frac{1}{2}}(\mathbf{x}, \mathbf{y})$  and  $\partial_{zw} D_{\frac{1}{2}}(\mathbf{x}, \mathbf{y})$ , for any  $\kappa > 0$  one has

$$\begin{aligned} D_{\pm\frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) &= D_{\pm\frac{1}{2}}(\mathbf{x}, \mathbf{y}) , \\ \partial_{zw} D_{\frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) &= \partial_{zw} D_{\frac{1}{2}}(\mathbf{x}, \mathbf{y}) . \end{aligned} \quad (2.24)$$

In the sequel we will consider the total energy VEV and express it in terms of the trace  $\text{Tr } \mathcal{A}^{-s}$ , fulfilling

$$\text{Tr } \mathcal{A}^{-s} = \int_{\Omega} d\mathbf{x} D_s(\mathbf{x}, \mathbf{x}) . \quad (2.25)$$

**2.6 The heat and cylinder kernels.** For  $\mathfrak{t} \in [0, +\infty)$ , these are given by

$$K(\mathfrak{t}; \mathbf{x}, \mathbf{y}) := e^{-\mathfrak{t}\mathcal{A}}(\mathbf{x}, \mathbf{y}) = \int_{\mathcal{K}} dk e^{-\mathfrak{t}\omega_k^2} F_k(\mathbf{x}) \overline{F_k}(\mathbf{y}) ; \quad (2.26)$$

$$T(\mathfrak{t}; \mathbf{x}, \mathbf{y}) := e^{-\mathfrak{t}\sqrt{\mathcal{A}}}(\mathbf{x}, \mathbf{y}) = \int_{\mathcal{K}} dk e^{-\mathfrak{t}\omega_k} F_k(\mathbf{x}) \overline{F_k}(\mathbf{y}) . \quad (2.27)$$

Sometimes we also consider the modified cylinder kernel

$$\tilde{T}(\mathfrak{t}; \mathbf{x}, \mathbf{y}) := (\sqrt{\mathcal{A}}^{-1} e^{-\mathfrak{t}\sqrt{\mathcal{A}}})(\mathbf{x}, \mathbf{y}) = \int_{\mathcal{K}} \frac{dk}{\omega_k} e^{-\mathfrak{t}\omega_k} F_k(\mathbf{x}) \overline{F_k}(\mathbf{y}) ; \quad (2.28)$$

it turns out that  $T(\mathfrak{t}; \mathbf{x}, \mathbf{y}) = -\partial_{\mathfrak{t}} \tilde{T}(\mathfrak{t}; \mathbf{x}, \mathbf{y})$ .

If  $\mathcal{A} = -\Delta + V$  ( $V$  smooth) is strictly positive, the map  $K(\mathfrak{t}; \cdot, \cdot) : \Omega \times \Omega \rightarrow \mathbf{C}$ ,  $(\mathbf{x}, \mathbf{y}) \mapsto K(\mathfrak{t}; \mathbf{x}, \mathbf{y})$  is continuous along with all its partial derivatives of any order, for all  $\mathfrak{t} > 0$  (the same holds for  $T$  and  $\tilde{T}$ ); with stronger assumptions on  $\mathcal{A}$ , the eigenfunction expansions in Eq.s (2.26) (2.27) (2.28) converge absolutely and uniformly, along with all their derivatives.

If  $\mathcal{A}$  is not strictly positive, but non-negative, Eq.s (2.26) (2.27) continue to make sense; moreover, if 0 is in the continuous spectrum of  $\mathcal{A}$ , its spectral measure vanishes and the operator  $\sqrt{\mathcal{A}}^{-1} e^{-\mathfrak{t}\sqrt{\mathcal{A}}}$  can still be defined; in this case, typically, the modified cylinder kernel and its eigenfunction expansion in (2.28) still make sense.

The *cylinder trace*, if it exists, is

$$T(\mathfrak{t}) := \text{Tr } e^{-\mathfrak{t}\sqrt{\mathcal{A}}} = \int_{\Omega} d\mathbf{x} T(\mathfrak{t}; \mathbf{x}, \mathbf{x}) . \quad (2.29)$$

**2.7 The Dirichlet kernel as Mellin transform of the heat or cylinder kernel.** For suitable values of  $s \in \mathbf{C}$  (see Part I), there hold

$$D_s(\mathbf{x}, \mathbf{y}) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt \, t^{s-1} K(t; \mathbf{x}, \mathbf{y}) ; \quad (2.30)$$

$$D_s(\mathbf{x}, \mathbf{y}) = \frac{1}{\Gamma(2s)} \int_0^{+\infty} dt \, t^{2s-1} T(t; \mathbf{x}, \mathbf{y}) . \quad (2.31)$$

Similar results hold for  $\text{Tr } \mathcal{A}^{-s}$ ; for example, using the cylinder trace  $T(t)$  of Eq. (2.29), we obtain

$$\text{Tr } \mathcal{A}^{-s} = \frac{1}{\Gamma(2s)} \int_0^{+\infty} dt \, t^{2s-1} T(t) . \quad (2.32)$$

**2.8 Analytic continuation of  $D_s$  via complex integration.** Let us assume

$$T(t; \mathbf{x}, \mathbf{y}) = \frac{1}{t^q} J(t; \mathbf{x}, \mathbf{y}) , \quad (2.33)$$

for some  $q \in \mathbf{Z}$ , where the map  $J : [0, +\infty) \times \Omega \times \Omega \rightarrow \mathbf{R}$  admits an extension  $J : \mathcal{U}([0, +\infty)) \times \Omega \times \Omega \rightarrow \mathbf{C}$  ( $\mathcal{U}([0, +\infty)) \subset \mathbf{C}$  is an open neighbourhood of  $[0, +\infty)$ ); besides, for fixed  $\mathbf{x}, \mathbf{y} \in \Omega$ , assume the function  $t \in \mathcal{U}([0, +\infty)) \mapsto J(t; \mathbf{x}, \mathbf{y})$  to be analytic and exponentially vanishing for  $\Re t \rightarrow +\infty$ . Then, one can infer from Eq. (2.31) that

$$D_s(\mathbf{x}, \mathbf{y}) = \frac{e^{-2i\pi s} \Gamma(1-2s)}{2\pi i} \int_{\mathfrak{H}} dt \, t^{2s-1} T(t; \mathbf{x}, \mathbf{y}) \quad (2.34)$$

where  $\mathfrak{H}$  denotes the *Hankel contour*, that is a simple path in the complex plane that starts in the upper half-plane near  $+\infty$ , encircles the origin counterclockwise and returns to  $+\infty$  in the lower half-plane (see Part I for an illustration of  $\mathfrak{H}$  and for a precise definition of the complex powers of  $t$ ).

Eq. (2.34) gives the analytic continuation of  $D_s$  to a meromorphic function of  $s$  on the whole complex plane, possibly with simple poles for  $s \in \{q/2, (q-1)/2, (q-2)/2, \dots\} \setminus \{0, -1/2, -1, -3/2, \dots\}$ . For half-integer values of  $s$ , the integral in Eq. (2.34) can be computed via the residue theorem; in this way for example, for  $s = -n/2$  and  $n \in \{0, 1, 2, \dots\}$ , we obtain

$$D_{-n/2}(\mathbf{x}, \mathbf{y}) = (-1)^n \Gamma(n+1) \text{Res}\left(t^{-(n+1)} T(t; \mathbf{x}, \mathbf{y}); 0\right) . \quad (2.35)$$

Using the modified cylinder kernel  $\tilde{T}$ , we deduce a variant of Eq. (2.34):

$$D_s(\mathbf{x}, \mathbf{y}) = - \frac{e^{-2i\pi s} \Gamma(2-2s)}{2\pi i} \int_{\mathfrak{H}} dt \, t^{2s-2} \tilde{T}(t; \mathbf{x}, \mathbf{y}) . \quad (2.36)$$

Again, for half-integer  $s$  the above analytic continuation can be computed explicitly by the residue theorem; more precisely, for  $n \in \{-1, 0, 1, 2, \dots\}$ , Eq. (2.36) gives

$$D_{-\frac{n}{2}}(\mathbf{x}, \mathbf{y}) = (-1)^{n+1} \Gamma(n+2) \operatorname{Res} \left( \mathfrak{t}^{-(n+2)} \tilde{T}(\mathfrak{t}; \mathbf{x}, \mathbf{y}); 0 \right) . \quad (2.37)$$

Similar results hold for the spatial derivatives of  $D_s$ ; moreover an analogous discussion can be made for the trace  $\operatorname{Tr} \mathcal{A}^{-s}$  starting from Eq. (2.32) and using the cylinder trace  $T(\mathfrak{t})$ . Assuming the latter to admit a meromorphic extension to a neighborhood of  $[0, +\infty)$ , with only a pole at  $\mathfrak{t} = 0$  and vanishing exponentially for  $\Re \mathfrak{t} \rightarrow +\infty$ , we have

$$\operatorname{Tr} \mathcal{A}^{-s} = \frac{e^{-2i\pi s} \Gamma(1-2s)}{2\pi i} \int_{\mathfrak{J}} d\mathfrak{t} \mathfrak{t}^{2s-1} T(\mathfrak{t}) . \quad (2.38)$$

For  $s = -n/2$  and  $n \in \{0, 1, 2, \dots\}$ , the above relation and the residue theorem give

$$\operatorname{Tr} \mathcal{A}^{n/2} = (-1)^n \Gamma(n+1) \operatorname{Res} \left( \mathfrak{t}^{-(n+1)} T(\mathfrak{t}); 0 \right) . \quad (2.39)$$

**2.9 The case of product domains. Factorization of the heat kernel.** Let  $\mathcal{A} := -\Delta + V$  and consider the case where

$$\Omega = \Omega_1 \times \Omega_2 \ni \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2), \mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) , \quad (2.40)$$

$$V(\mathbf{x}) = V_1(\mathbf{x}_1) + V_2(\mathbf{x}_2) \quad (2.41)$$

( $\Omega_a \subset \mathbf{R}^{d_a}$  is an open subset, for  $a \in \{1, 2\}$ ;  $d_1 + d_2 = d$ ); assume the boundary conditions on  $\partial\Omega$  to arise from suitable boundary conditions prescribed separately on  $\partial\Omega_1$  and  $\partial\Omega_2$  so that, for  $a = 1, 2$ , the operators

$$\mathcal{A}_a := -\Delta_a + V(\mathbf{x}_a) \quad (2.42)$$

(with  $\Delta_a$  the Laplacian on  $\Omega_a$ ) are selfadjoint and strictly positive (or at least, non-negative) in  $L^2(\Omega_a)$ . Then, the Hilbert space  $L^2(\Omega)$  and the operator  $\mathcal{A}$  can be represented as

$$L^2(\Omega) = L^2(\Omega_1) \otimes L^2(\Omega_2) , \quad \mathcal{A} = \mathcal{A}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{A}_2 . \quad (2.43)$$

This implies, amongst else, that the heat kernels  $K(\mathfrak{t}; \mathbf{x}, \mathbf{y}) := e^{\mathfrak{t}\mathcal{A}}(\mathbf{x}, \mathbf{y})$ ,  $K_a(\mathfrak{t}; \mathbf{x}_a, \mathbf{y}_a) := e^{\mathfrak{t}\mathcal{A}_a}(\mathbf{x}_a, \mathbf{y}_a)$  ( $a = 1, 2$ ) are related by

$$K(\mathfrak{t}; \mathbf{x}, \mathbf{y}) = K_1(\mathfrak{t}; \mathbf{x}_1, \mathbf{y}_1) K_2(\mathfrak{t}; \mathbf{x}_2, \mathbf{y}_2) . \quad (2.44)$$

## 2.10 The subcase of a slab: reduction to a lower-dimensional problem.

By definition, we have a *slab* if

$$\Omega = \Omega_1 \times \mathbf{R}^{d_2} \ni \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2), \mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2), \quad V(\mathbf{x}) = V(\mathbf{x}_1) \quad (2.45)$$

with  $\Omega_1 \subset \mathbf{R}^{d_1}$  an open subset ( $d_1 + d_2 = d$ ), and if the boundary conditions prescribed for the field only refer to  $\partial\Omega_1 \times \mathbf{R}^{d_2}$ . We write  $D_s(\mathbf{x}, \mathbf{y})$  for the Dirichlet kernel of  $\mathcal{A} := -\Delta + V(\mathbf{x}_1)$  at  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ ,  $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in \Omega$ ;  $D_s^{(1)}(\mathbf{x}_1, \mathbf{y}_1)$  is the Dirichlet kernel of  $\mathcal{A}_1 := -\Delta_1 + V(\mathbf{x}_1)$ . There hold

$$D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \frac{\Gamma(\frac{u-d_2+1}{2})}{(4\pi)^{d_2/2} \Gamma(\frac{u+1}{2})} D_{\frac{u-d_2+1}{2}}^{(1)}(\mathbf{x}_1, \mathbf{y}_1) \Big|_{\mathbf{y}_1=\mathbf{x}_1}; \quad (2.46)$$

$$\partial_{x_a^i y_b^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \partial_{x_a^i x_b^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \partial_{y_a^i y_b^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = 0 \quad (2.47)$$

for  $(a, b) = (1, 2)$  or  $(a, b) = (2, 1)$  and  $i \in \{1, \dots, d_a\}$ ,  $j \in \{1, \dots, d_b\}$ ;

$$\partial_{z_1^i w_1^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \frac{\Gamma(\frac{u-d_2+1}{2})}{(4\pi)^{d_2/2} \Gamma(\frac{u+1}{2})} \partial_{z_1^i w_1^j} D_{\frac{u-d_2+1}{2}}^{(1)}(\mathbf{x}_1, \mathbf{y}_1) \Big|_{\mathbf{y}_1=\mathbf{x}_1} \quad (2.48)$$

for  $z, w \in \{x, y\}$  and  $i, j \in \{1, \dots, d_1\}$ ;

$$\begin{aligned} \partial_{x_2^i y_2^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} &= -\partial_{x_2^i x_2^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = -\partial_{y_2^i y_2^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \\ &= \delta_{ij} \frac{\Gamma(\frac{u-d_2-1}{2})}{(4\pi)^{d_2/2} 2 \Gamma(\frac{u+1}{2})} D_{\frac{u-d_2-1}{2}}^{(1)}(\mathbf{x}_1, \mathbf{y}_1) \Big|_{\mathbf{y}_1=\mathbf{x}_1} \quad \text{for } i, j \in \{1, \dots, d_2\}. \end{aligned} \quad (2.49)$$

The above relations, along with Eq.s (2.16-2.18), imply

$$\begin{aligned} \langle 0 | \hat{T}_{ij}^u(\mathbf{x}) | 0 \rangle &= 0 \quad \text{for } i, j \in \{d_1+1, \dots, d\}, i \neq j; \\ \langle 0 | \hat{T}_{ij}^u(\mathbf{x}) | 0 \rangle &= \langle 0 | \hat{T}_{ji}^u(\mathbf{x}) | 0 \rangle = 0 \quad \text{for } i \in \{1, \dots, d_1\}, j \in \{d_1+1, \dots, d\}. \end{aligned} \quad (2.50)$$

Clearly, analogous relations hold for the renormalized VEV  $\langle 0 | \hat{T}_{ij}^u(\mathbf{x}) | 0 \rangle_{ren}$ .

**2.11 Reduced energy for a slab configuration.** Consider the slab configuration of subsection 2.10; the *reduced regularized energy* (i.e., the total energy per unit volume in the “free” dimensions) is

$$\mathcal{E}_1^u := \int_{\Omega_1} d\mathbf{x}_1 \langle 0 | \hat{T}_{00}^u | 0 \rangle = E_1^u + B_1^u. \quad (2.51)$$

The second equality is proved after defining the *regularized reduced bulk* and *boundary energies*, which are

$$\begin{aligned} E_1^u &:= \frac{\kappa^u \Gamma(\frac{u-d_2-1}{2})}{2 (4\pi)^{d_2/2} \Gamma(\frac{u-1}{2})} \int_{\Omega_1} d\mathbf{x}_1 D_{\frac{u-d_2-1}{2}}^{(1)}(\mathbf{x}_1, \mathbf{x}_1) = \\ &= \frac{\kappa^u \Gamma(\frac{u-d_2-1}{2})}{2 (4\pi)^{d_2/2} \Gamma(\frac{u-1}{2})} \text{Tr } \mathcal{A}_1^{\frac{d_2+1-u}{2}}, \end{aligned} \quad (2.52)$$

$$B_1^u := \frac{\kappa^u \Gamma(\frac{u-d_2+1}{2})}{(4\pi)^{d_2/2} \Gamma(\frac{u+1}{2})} \left( \frac{1}{4} - \xi \right) \int_{\partial\Omega_1} da(\mathbf{x}_1) \frac{\partial}{\partial n_{\mathbf{y}_1}} D_{\frac{u-d_2+1}{2}}^{(1)}(\mathbf{x}_1, \mathbf{y}_1) \Big|_{\mathbf{y}_1=\mathbf{x}_1} . \quad (2.53)$$

One has  $B_1^u = 0$  for  $\Omega_1$  bounded and either Dirichlet or Neumann boundary conditions on  $\partial\Omega_1$ . Assuming the functions (2.52) (2.53) to be finite and analytic for suitable  $u \in \mathbf{C}$ , we define the renormalized, reduced bulk energy by the generalized (or restricted) zeta approach:

$$E_1^{ren} := RP \Big|_{u=0} E_1^u \quad \left( \text{or } E_1^{ren} := E_1^u \Big|_{u=0} \right) . \quad (2.54)$$

**2.12 Pressure on the boundary.** This is the force per unit area produced by the quantized field inside  $\Omega$  at a point  $\mathbf{x} \in \partial\Omega$ . We first consider, for  $\Re u$  large, the *regularized pressure*  $\mathbf{p}^u(\mathbf{x})$  with components

$$p_i^u(\mathbf{x}) := \langle 0 | \widehat{T}_{ij}^u(\mathbf{x}) | 0 \rangle n^j(\mathbf{x}) ; \quad (2.55)$$

here and in the remainder of this paper,  $\mathbf{n}(\mathbf{x}) \equiv (n^i(\mathbf{x}))$  is the unit outer normal at  $\mathbf{x} \in \partial\Omega$ . For Dirichlet boundary conditions the above definition implies

$$p_i^u(\mathbf{x}) = \kappa^u \left[ \left( -\frac{1}{4} \delta_{ij} \partial^{x^\ell} \partial_{y^\ell} + \frac{1}{2} \partial_{x^i y^j} \right) D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \right]_{\mathbf{y}=\mathbf{x}} n^j(\mathbf{x}) . \quad (2.56)$$

We can define the *renormalized pressure* by analytic continuation as

$$p_i^{ren}(\mathbf{x}) := RP \Big|_{u=0} p_i^u(\mathbf{x}) \quad (2.57)$$

(first compute the regularized pressure at the boundary, and then analytically continue at  $u = 0$ ); alternatively, we could put

$$p_i^{ren}(\mathbf{x}) := \left( \lim_{\mathbf{x}' \in \Omega, \mathbf{x}' \rightarrow \mathbf{x}} \langle 0 | \widehat{T}_{ij}(\mathbf{x}') | 0 \rangle_{ren} \right) n^j(\mathbf{x}) \quad (2.58)$$

(first renormalize the stress-energy VEV at inner points of  $\Omega$ , and then move to the boundary).

Prescriptions (2.57) (2.58) do not always agree (for a counterexample, see Section 5). In Part I we conjectured that the two approaches agree when both of them give a finite result (this is true in all the examples of the present Part II).

**2.13 The Hilbert space when 0 is an isolated point of  $\sigma(\mathcal{A})$ ; the case of Neumann and periodic boundary conditions.** Assume  $\mathcal{A} = -\Delta + V$ , acting on  $L^2(\Omega)$ , to have its spectrum contained in  $[0, +\infty)$ , with 0 an isolated point (hence a proper eigenvalue); in this case, we replace the basic Hilbert space  $L^2(\Omega)$  with

$$L_0^2(\Omega) := (\ker \mathcal{A})^\perp \quad (\subset L^2(\Omega)) . \quad (2.59)$$

The restriction of  $\mathcal{A}$  to  $L_0^2(\Omega)$  is selfadjoint and strictly positive; we take  $L_0^2(\Omega)$  as the basic space even for the field quantization.

For example, if  $\mathcal{A} = -\Delta$ ,  $\Omega$  is bounded and either Neumann or periodic boundary conditions are prescribed on  $\partial\Omega$ , one finds

$$L_0^2(\Omega) = \left\{ f \in L^2(\Omega) \mid \int_{\Omega} d\mathbf{x} f(\mathbf{x}) = 0 \right\}. \quad (2.60)$$

For slab configurations where  $\Omega = \Omega_1 \times \mathbf{R}^{d_2}$  and Neumann or periodic boundary conditions are prescribed on  $\partial\Omega_1 \times \mathbf{R}^{d_2}$ , we set ( $\mathcal{A}_1$  is the reduced operator in  $L^2(\Omega_1)$ )

$$L_0^2(\Omega_1) := (\ker \mathcal{A}_1)^\perp = \left\{ f \in L^2(\Omega_1) \mid \int_{\Omega_1} d\mathbf{x}_1 f(\mathbf{x}_1) = 0 \right\}; \quad (2.61)$$

the basic Hilber space for the full theory on  $\Omega$  is  $L_0^2(\Omega_1) \otimes L^2(\mathbf{R}^{d_2})$ .

**2.14 The case where 0 is in the continuous spectrum of  $\mathcal{A}$ .** Assume the fundamental operator  $\mathcal{A} = -\Delta + V$  to be non-negative ( $\sigma(\mathcal{A}) \subset [0, +\infty)$ ), with 0 in the continuous spectrum of  $\mathcal{A}$ . The approach we consider in this case is to represent  $\mathcal{A}$  as

$$\mathcal{A} := \text{“} \lim_{\varepsilon \rightarrow 0^+} \text{”} \mathcal{A}_\varepsilon \quad (2.62)$$

where, for  $\varepsilon \in (0, \varepsilon_0)$ ,  $\mathcal{A}_\varepsilon$  is a selfadjoint operator in  $L^2(\Omega)$  with  $\sigma(\mathcal{A}_\varepsilon) \subset [\varepsilon^2, +\infty)$ . We define a *deformed, smeared field operator*  $\hat{\phi}^{\varepsilon u} := (\kappa^{-2} \mathcal{A}_\varepsilon)^{-u/4} \hat{\phi}$  and a *deformed, regularized stress-energy tensor operator*  $\hat{T}_{\mu\nu}^{\varepsilon u}$ , whose VEV is

$$\begin{aligned} \langle 0 | \hat{T}_{\mu\nu}^{\varepsilon u}(x) | 0 \rangle &= \\ &= \left( \frac{1}{2} - \xi \right) (\partial_{x^\mu} y^\nu + \partial_{x^\nu} y^\mu) - \left( \frac{1}{2} - 2\xi \right) \eta_{\mu\nu} \left( \partial^{x^\lambda} \partial_{y^\lambda} + V \right) - \xi (\partial_{x^\mu} x^\nu + \partial_{y^\mu} y^\nu) \Big|_{y=x} \\ &\cdot \langle 0 | \hat{\phi}^{\varepsilon u}(x) \hat{\phi}^{\varepsilon u}(y) | 0 \rangle; \end{aligned} \quad (2.63)$$

similarly, for  $\mathbf{x} \in \partial\Omega$ , we consider the *deformed, regularized pressure*

$$p_i^{\varepsilon u}(\mathbf{x}) := \langle 0 | \hat{T}_{ij}^{\varepsilon u}(\mathbf{x}) | 0 \rangle n^j(\mathbf{x}). \quad (2.64)$$

In the end, we put

$$\langle 0 | \hat{T}_{\mu\nu}(x) | 0 \rangle_{ren} := \lim_{\varepsilon \rightarrow 0^+} RP \Big|_{u=0} \langle 0 | \hat{T}_{\mu\nu}^{\varepsilon u}(x) | 0 \rangle; \quad (2.65)$$

$$p_i^{ren}(\mathbf{x}) := \lim_{\varepsilon \rightarrow 0^+} RP \Big|_{u=0} p_i^{\varepsilon u}(\mathbf{x}). \quad (2.66)$$

We could alternatively put

$$p_i^{ren}(\mathbf{x}) := \left( \lim_{\mathbf{x}' \in \Omega, \mathbf{x}' \rightarrow \mathbf{x}} \langle 0 | \widehat{T}_{ij}(\mathbf{x}') | 0 \rangle_{ren} \right) n^j(\mathbf{x}) \quad (2.67)$$

( $\langle 0 | \widehat{T}_{ij}(\mathbf{x}') | 0 \rangle_{ren}$  is defined via Eq. (2.65)). For the VEV (2.63) we have an expression analogous to (2.16-2.18) in terms of the *deformed Dirichlet kernel*

$$D_s^\varepsilon(\mathbf{x}, \mathbf{y}) := \mathcal{A}_\varepsilon^s(\mathbf{x}, \mathbf{y}) = \langle \delta_{\mathbf{x}} | \mathcal{A}_\varepsilon^s \delta_{\mathbf{y}} \rangle, \quad (2.68)$$

with  $s = (u \pm 1)/2$ . For the renormalized stress-energy VEV (2.65) we have an expression of the form (2.21-2.23), where

$$D_{\pm \frac{1}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) := \lim_{\varepsilon \rightarrow 0^+} RP \Big|_{u=0} \left( \kappa^u D_{\frac{u \pm 1}{2}}^\varepsilon(\mathbf{x}, \mathbf{y}) \right) = \lim_{\varepsilon \rightarrow 0^+} RP \Big|_{s=\pm \frac{1}{2}} \left( \kappa^{2s \mp 1} D_s^\varepsilon(\mathbf{x}, \mathbf{y}) \right), \quad (2.69)$$

and analogous definitions hold for the spatial derivatives in the cited equations.

In Part I we showed that two useful choices of  $\mathcal{A}_\varepsilon$  are the following ( $K^\varepsilon, T^\varepsilon$  and  $K, T$  denote the heat and cylinder kernels associated to  $\mathcal{A}_\varepsilon$  and  $\mathcal{A}$ , respectively):

$$\mathcal{A}_\varepsilon := \mathcal{A} + \varepsilon^2 \quad \Rightarrow \quad K^\varepsilon(\mathbf{t}; \mathbf{x}, \mathbf{y}) = e^{-\varepsilon^2 \mathbf{t}} K(\mathbf{t}; \mathbf{x}, \mathbf{y}); \quad (2.70)$$

$$\mathcal{A}_\varepsilon := (\sqrt{\mathcal{A}} + \varepsilon)^2 \quad \Rightarrow \quad T^\varepsilon(\mathbf{t}; \mathbf{x}, \mathbf{y}) = e^{-\varepsilon \mathbf{t}} T(\mathbf{t}; \mathbf{x}, \mathbf{y}). \quad (2.71)$$

Let  $n \in \{-1, 0, 1, 2, \dots\}$ . If the modified cylinder kernel  $\tilde{T}(\mathbf{t}; \mathbf{x}, \mathbf{y})$  of  $\mathcal{A}$  admits a meromorphic extension in  $\mathbf{t}$  to a neighborhood of  $[0, +\infty)$  fulfilling

$$|\tilde{T}(\mathbf{t}; \mathbf{x}, \mathbf{y})| \leq C |\mathbf{t}|^{-a-n+1} \quad \text{for } \Re \mathbf{t} \rightarrow +\infty \text{ and some } C, a > 0, \quad (2.72)$$

then, assuming  $\mathcal{A}_\varepsilon$  to be as in Eq. (2.71), we have

$$\begin{aligned} D_{-\frac{n}{2}}^{(\kappa)}(\mathbf{x}, \mathbf{y}) &:= \lim_{\varepsilon \rightarrow 0^+} RP \Big|_{s=-\frac{n}{2}} \left( \kappa^{2s+n} D_s^\varepsilon(\mathbf{x}, \mathbf{y}) \right) = \\ &(-1)^{n+1} \Gamma(n+2) \text{Res} \left( \mathbf{t}^{-(n+2)} \tilde{T}(\mathbf{t}; \mathbf{x}, \mathbf{y}); 0 \right). \end{aligned} \quad (2.73)$$

With analogous hypotheses for  $\partial_{zw} \tilde{T}(\mathbf{t}; \mathbf{x}, \mathbf{y})$  ( $z, w$  any pair of spatial variables), similar relations for the spatial derivatives can be deduced.

**2.15 Some variations involving the spatial domain.** Configurations involving periodic boundary conditions are formulated rigorously describing  $\Omega$  in terms of tori; e.g., the domain  $\Omega = (0, a)^d$  with periodic boundary conditions is viewed as the torus  $\mathbf{T}_a^d := \mathbf{R}^d / (a\mathbf{Z})^d \simeq (\mathbf{R}/a\mathbf{Z})^d$  <sup>(3)</sup>.

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<sup>3</sup>The considerations of subsection 2.13 for the periodic case are easily rephrased in terms of the torus  $\mathbf{T}_a^d$  (see footnote 21 in Part I).



Sometimes (see Section 5) we employ on  $\Omega$  some set of curvilinear coordinates  $(q^i)_{i=1,\dots,d} \equiv \mathbf{q}$ , inducing a set of spacetime coordinates  $q \equiv (q^\mu) \equiv (t, \mathbf{q})$ ; the spatial and space-time line elements are, respectively

$$\begin{aligned} d\ell^2 &= a_{ij}(\mathbf{q}) dq^i dq^j ; & ds^2 &= -dt^2 + d\ell^2 = g_{\mu\nu}(q) dq^\mu dq^\nu , \\ g_{00} &:= -1 , & g_{i0} = g_{0i} &:= 0 , & g_{ij}(q) &:= a_{ij}(\mathbf{q}) \quad \text{for } i, j \in \{1, \dots, d\} . \end{aligned} \quad (2.74)$$

The analogue of Eq. (2.8) in the coordinate system  $(q^\mu)$  is

$$\hat{T}_{\mu\nu}^u := (1-2\xi) \partial_\mu \hat{\phi}^u \circ \partial_\nu \hat{\phi}^u - \left( \frac{1}{2} - 2\xi \right) \eta_{\mu\nu} \left( \partial^\lambda \hat{\phi}^u \partial_\lambda \hat{\phi}^u + V(\hat{\phi}^u)^2 \right) - 2\xi \hat{\phi}^u \circ \nabla_{\mu\nu} \hat{\phi}^u \quad (2.75)$$

( $\nabla_\mu$  is the covariant derivative induced by the metric (2.74)). For any scalar function  $f$  there hold ( $\gamma_{ij}^k$  are the Christoffel symbols for the spatial metric  $(a_{ij}(\mathbf{q}))$ )

$$\begin{aligned} \nabla_\mu f &= \partial_\mu f , & \nabla_{ij} f &= D_{ij} f = \partial_{ij} f - \gamma_{ij}^k \partial_k f , \\ \nabla_{0i} f &= \partial_0(\partial_i f) = \partial_i(\partial_0 f) = \nabla_{i0} f , & \nabla_{00} f &= \partial_{00} f . \end{aligned} \quad (2.76)$$

Many results in the previous subsections are readily adapted to the variations considered in this subsection for the space domain.

### 3 The case of a massless field between parallel hyperplanes

**3.1 Introducing the problem for arbitrary boundary conditions.** As mentioned in Section 6 of Part I, the segment configuration can be considered as the  $d = 1$  case of a general,  $d$ -dimensional configuration with two parallel hyperplanes; this is the subject we are now going to analyze (with no external potential). So, we assume

$$\Omega := (0, a) \times \mathbf{R}^{d-1} , \quad a > 0 , \quad V = 0 ; \quad (3.1)$$

these choices correspond to a massless scalar field confined between the two parallel hyperplanes <sup>(4)</sup>

$$\pi_0 = \{\mathbf{x} \in \mathbf{R}^d \mid x^1 = 0\} , \quad \pi_a = \{\mathbf{x} \in \mathbf{R}^d \mid x^1 = a\} . \quad (3.2)$$

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<sup>4</sup> There holds comments analogous to the ones in footnote 18 on page 49 of Part I; namely, in place of the standard Cartesian coordinates  $\mathbf{x} \equiv (x^i)$ , we could have used the set of rescaled coordinates (best fitting the features of the present configuration)

$$\mathbf{x}_\star \equiv (x_\star^i)_{i=1,\dots,d} , \quad \text{with } x_\star^1 := x^1/a , \quad x_\star^i := x^i \quad \text{for } i \in \{2, \dots, d\} .$$

Also in this case, we choose not to employ the above rescaled coordinate system in order to render comparison with known results easier.

As we did in Section 6 of Part I for the segment configuration, we are going to consider separately the cases where the field fulfills Dirichlet, Neumann or periodic boundary conditions on the hyperplanes  $\pi_0, \pi_a$ . Throughout this section we assume

$$d \text{ odd , } d \geq 3 ; \quad (3.3)$$

this hypothesis is purely technical and will be motivated later (see the comments after Eq. (3.6)); note that the case  $d = 1$ , here excluded, has been already discussed in Part I.

In passing let us notice that, for  $d = 3$  and Dirichlet boundary conditions (see subsection 3.6), the above configuration is the one most typically considered when dealing with the (scalar) Casimir effect [4, 25, 30]. The case with Dirichlet boundary conditions on one plane and Neumann conditions on the other (discussed in subsection 3.7) was originally considered in the electromagnetic case by Boyer [5], who derived the total energy; later, computations of the total energy and boundary forces for a (massless or massive) scalar field at both zero and non-zero temperature were performed by Pinto et al. [10, 28] and Santos et al. [32] (see also [2, 4]). Finally, let us also mention the monography by Fulling [22] where the stress-energy VEV for the model with periodic boundary conditions is given; see also [14, 15] and, again, [4, 25] for the derivation of the total energy in the same configuration.

For any one of the boundary conditions mentioned above, keeping in mind the considerations of subsection 2.13, we proceed in the manner explained in the following subsection. Before moving on, let us remark that, due to the results on slab configurations reported in subsection 2.10, we just have to study the reduced one-dimensional problem based on

$$\Omega_1 := (0, a) \subset \mathbf{R} , \quad \mathcal{A}_1 := -\partial_{x^1 x^1} , \quad (3.4)$$

keeping into account the boundary conditions in  $x^1 = 0$  and  $x^1 = a$ ; in consequence of this, we can resort to the results of Section 6 in Part I.

In subsections 3.2-3.5 we will present some general results on the configuration under analysis, holding for all the types of boundary conditions mentioned before. In subsections 3.6-3.9 we will consider specific boundary conditions, with a special attention for the case  $d = 3$ .

**3.2 The reduced Dirichlet and cylinder kernels.** According to Eq.s (2.46-2.49), the basic ingredients for the analysis of the  $d$ -dimensional problem are the Dirichlet kernel  $D_s^{(1)}$  of the reduced 1-dimensional problem at  $s = (u - d)/2$ , and its derivatives at  $s = (u - d + 2)/2$ . On the other hand, these functions can be expressed in terms of the 1-dimensional cylinder kernel  $T^{(1)}(\mathbf{t}; x^1, y^1)$ , which has been determined in Section 6 of Part I (where it was indicated simply with  $T(\mathbf{t}; x^1, y^1)$ ); let us recall that this kernel and all its derivatives, when evaluated on the diagonal

$y^1 = x^1$ , are meromorphic functions of  $\mathbf{t}$  in a neighborhood of the positive real half-axis with a unique singularity in  $\mathbf{t} = 0$ , and they vanish exponentially for  $\Re \mathbf{t} \rightarrow +\infty$ . For the reduced Dirichlet kernel and for the derivatives in which we are interested, Eq. (2.34) yields the following expressions:

$$D_{\frac{u-d}{2}}^{(1)}(x^1, y^1) = \frac{e^{-i\pi(u-d)} \Gamma(d+1-u)}{2\pi i} \int_{\mathfrak{H}} d\mathbf{t} \, \mathbf{t}^{u-d-1} T^{(1)}(\mathbf{t}; x^1, y^1) ; \quad (3.5)$$

$$\partial_{zw} D_{\frac{u-d+2}{2}}^{(1)}(x^1, y^1) = \frac{e^{-i\pi(u-d)} \Gamma(d-1-u)}{2\pi i} \int_{\mathfrak{H}} d\mathbf{t} \, \mathbf{t}^{u-d+1} \partial_{zw} T^{(1)}(\mathbf{t}; x^1, y^1) \quad (3.6)$$

for  $z, w \in \{x^1, y^1\}$  .

Consider the above relations along with Eq.s (2.46-2.49), relating the 1-dimensional functions to the  $d$ -dimensional Dirichlet kernel  $D_{\frac{u-1}{2}}(\mathbf{x}, \mathbf{y})$  or to the derivatives of  $D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y})$ ; these equations yield the sought-for meromorphic continuations in  $u$  to the whole complex plane. By direct inspection of the expressions thus obtained it appears that, with the assumption (3.3) on  $d$ , these meromorphic continuations are analytic for  $u$  in a neighborhood of the origin; so, the zeta strategy for renormalization is implemented by simply setting  $u = 0$ . In conclusion, we have the following integral representations for the renormalized Dirichlet kernel and for its renormalized derivatives:

$$D_{-\frac{1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = -\frac{C_d}{2\pi i} \int_{\mathfrak{H}} \frac{d\mathbf{t}}{\mathbf{t}^{d+1}} T^{(1)}(\mathbf{t}; x^1, y^1) \Big|_{y^1=x^1} ; \quad (3.7)$$

$$\partial_{zw} D_{\frac{1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = -\frac{C_d}{(d-1)2\pi i} \int_{\mathfrak{H}} \frac{d\mathbf{t}}{\mathbf{t}^{d-1}} \partial_{zw} T^{(1)}(\mathbf{t}; x^1, y^1) \Big|_{y^1=x^1} \text{ for } z, w \in \{x^1, y^1\} ; \quad (3.8)$$

$$\begin{aligned} \partial_{x_2^i y_2^j} D_{\frac{1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} &= -\partial_{x_2^i x_2^j} D_{\frac{1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = -\partial_{y_2^i y_2^j} D_{\frac{1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \\ &= \delta_{ij} \frac{C_d}{2\pi i} \int_{\mathfrak{H}} \frac{d\mathbf{t}}{\mathbf{t}^{d+1}} T^{(1)}(\mathbf{t}; x^1, y^1) \Big|_{y^1=x^1} \text{ for } i, j \in \{1, \dots, d-1\} ; \end{aligned} \quad (3.9)$$

for the sake of brevity, in the above we have set <sup>(5)</sup>

$$C_d := (-\pi)^{-\frac{d-1}{2}} \Gamma\left(\frac{d+1}{2}\right) . \quad (3.10)$$

Eq.s (3.7-3.9) are completed with Eq. (2.47), stating the vanishing of certain mixed derivatives.

Finally, recall that Eq. (2.52) (here employed with  $d_2 = d-1$ ) for the regularized, reduced bulk energy (see the subsequent subsection 3.4) requires the evaluation of

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<sup>5</sup>To justify the expression (3.10), some identities regarding the gamma function must be used.

the trace  $\text{Tr } \mathcal{A}_1^{\frac{d-u}{2}}$ . To this purpose, we first consider the one-dimensional cylinder trace  $T^{(1)}(\mathbf{t})$ , which has also been determined in Section 6 of Part I (where it was indicated simply with  $T(\mathbf{t})$ ); we recall that, for all the boundary conditions considered in the following applications, the map  $\mathbf{t} \mapsto T^{(1)}(\mathbf{t})$  admits a meromorphic extension to a neighborhood of  $[0, +\infty)$  which only has a pole at  $\mathbf{t} = 0$  and vanishes exponentially for  $\Re \mathbf{t} \rightarrow +\infty$ .

A discussion similar to the one carried over above for the Dirichlet kernel allows us to derive an explicit expression for the analytic continuation of  $\text{Tr } \mathcal{A}_1^{\frac{d-u}{2}}$  at  $u = 0$ ; more precisely, using for the reduced problem a relation analogous to one in Eq. (2.38), we conclude

$$\text{Tr } \mathcal{A}_1^{d/2} = \frac{(-1)^d \Gamma(d+1)}{2\pi i} \int_{\mathfrak{H}} d\mathbf{t} \, \mathbf{t}^{-(d+1)} T^{(1)}(\mathbf{t}) . \quad (3.11)$$

**3.3 The stress-energy tensor.** Substituting the relations (2.47) and (3.7-3.9) into Eq.s (2.16-2.18), we straightforwardly deduce the contour integral representations for the non-vanishing components of the renormalized stress-energy VEV; moreover, due to the meromorphic nature of the cylinder kernel (and of its derivatives), the resulting integrals along the Hankel contour can be explicitly evaluated via the residue theorem. The final expressions for the (non-zero) components of the renormalized stress-energy VEV are the following ones:

$$\begin{aligned} \langle 0 | \widehat{T}_{00}(\mathbf{x}) | 0 \rangle_{ren} = & -C_d \text{Res} \left( \mathbf{t}^{-(d+1)} \left[ \left( \xi - \frac{d-2}{4d} \right) d T^{(1)}(\mathbf{t}; x^1, y^1) + \right. \right. \\ & \left. \left. + \frac{\mathbf{t}^2}{d-1} \left( \frac{1}{4} - \xi \right) \partial_{x^1 y^1} T^{(1)}(\mathbf{t}; x^1, y^1) \right]_{y^1=x^1}; 0 \right); \end{aligned} \quad (3.12)$$

$$\begin{aligned} \langle 0 | \widehat{T}_{11}(\mathbf{x}) | 0 \rangle_{ren} = & -C_d \text{Res} \left( \mathbf{t}^{-(d+1)} \left[ \left( \frac{1}{4} - \xi \right) d T^{(1)}(\mathbf{t}; x^1, y^1) + \right. \right. \\ & \left. \left. + \frac{\mathbf{t}^2}{d-1} \left( \frac{1}{4} \partial_{x^1 y^1} - \xi \partial_{x^1 x^1} \right) T^{(1)}(\mathbf{t}; x^1, y^1) \right]_{y^1=x^1}; 0 \right); \end{aligned} \quad (3.13)$$

$$\begin{aligned} \langle 0 | \widehat{T}_{ij}(\mathbf{x}) | 0 \rangle_{ren} = & \delta_{ij} C_d \text{Res} \left( \mathbf{t}^{-(d+1)} \left[ \left( \xi - \frac{d-2}{4d} \right) d T^{(1)}(\mathbf{t}; x^1, y^1) + \right. \right. \\ & \left. \left. + \frac{\mathbf{t}^2}{d-1} \left( \frac{1}{4} - \xi \right) \partial_{x^1 y^1} T^{(1)}(\mathbf{t}; x^1, y^1) \right]_{y^1=x^1}; 0 \right) \quad \text{for } i, j \in \{2, \dots, d\}. \end{aligned} \quad (3.14)$$

We repeat that, in the above,  $d$  is an arbitrary odd dimension  $> 1$ . Starting from subsection 3.6, for each one of the previously mentioned boundary conditions we will report the explicit expressions for the stress-energy components arising from Eq.s (3.12-3.14) in the case of spatial dimension  $d = 3$ ; again, we will give the final results

in the form described in subsection 2.3 (see, in particular, Eq. (2.13)), noting that (2.14) gives

$$\xi_3 = \frac{1}{6} . \quad (3.15)$$

Some additional details related to these computations will be given, as examples, in the case of Dirichlet and periodic boundary conditions.

**3.4 The reduced energy.** Let us recall again that this is the energy per unit volume in the free dimensions and that it can be expressed in terms of the reduced bulk and boundary energies (see subsection 2.11).

The general identity (2.52) allows us to represent the reduced bulk energy in terms of the renormalized trace  $\text{Tr } \mathcal{A}_1^{d/2}$ , corresponding to the reduced operator  $\mathcal{A}_1$ ; on the other hand, Eq. (3.11) gives an explicit expression for the latter quantity in terms of the reduced cylinder trace  $T^{(1)}(\mathfrak{t})$ . Evaluating the integral along the Hankel contour in the cited equation via the residue theorem (compare with Eq. (2.39)), we conclude

$$E_1^{ren} = \frac{(-1)^{d+1} \Gamma(d+1) \Gamma(-\frac{d}{2})}{2 (4\pi)^{d/2}} \text{Res} \left( \mathfrak{t}^{-(d+1)} T^{(1)}(\mathfrak{t}); 0 \right) . \quad (3.16)$$

Let us also mention that, for any one of the boundary conditions to be considered in the following, the (regularized and renormalized) reduced boundary energy always vanishes identically.

Finally we point out that, at least in spatial dimension  $d = 3$ , the results obtained via Eq. (3.16) for the renormalized, reduced bulk energy coincide with the integral over the interval  $(0, a)$  of the conformal part of the corresponding renormalized energy density  $\langle 0 | \widehat{T}_{00} | 0 \rangle_{ren}$ ; on the contrary, the non-conformal part of the latter appears to diverge in a non-integrable manner near the planes  $\pi_0, \pi_a$ . These facts closely resemble the ones pointed out for the total energy of the segment configuration in Section 6 of Part I; as in that case, they will be checked by direct computation in subsections 3.6-3.9.

**3.5 The boundary forces.** The situation we meet in the present situation is of the kind described in general in subsection 2.12, and already faced for the segment configuration in Part I: in principle, we can define the renormalized pressure on the plates  $\pi_0, \pi_a$  in two alternative ways.

Let  $\mathbf{n}(\mathbf{x})$  denote the unit “outer normal” at points on the boundary, so that  $\mathbf{n}(\mathbf{x}) = (-1, 0, \dots, 0)$  on  $\pi_0$  and  $\mathbf{n}(\mathbf{x}) = (1, 0, \dots, 0)$  on  $\pi_a$ . Then, on the one hand we can put

$$p_i^{ren}(\mathbf{x}) := \langle 0 | \widehat{T}_{ij}^u(\mathbf{x}) | 0 \rangle \Big|_{u=0} \quad n^j(\mathbf{x}) = \delta_{i1} \langle 0 | \widehat{T}_{11}^u(\mathbf{x}) | 0 \rangle \Big|_{u=0} ; \quad (3.17)$$

on the other hand, we have the alternative definition

$$\begin{aligned} p_i^{ren}(\mathbf{x}) &:= \left( \lim_{\mathbf{x}' \in \Omega, \mathbf{x}' \rightarrow \mathbf{x}} \langle 0 | \widehat{T}_{ij}(\mathbf{x}') | 0 \rangle_{ren} \right) n^j(\mathbf{x}) \\ &= \delta_{i1} \left( \lim_{\mathbf{x}' \in \Omega, \mathbf{x}' \rightarrow \mathbf{x}} \langle 0 | \widehat{T}_{11}(\mathbf{x}') | 0 \rangle_{ren} \right). \end{aligned} \quad (3.18)$$

As a matter of fact, definitions (3.17) and (3.18) will be found by explicit computations to yield the same result for any one of the boundary conditions to be considered in the next subsections.

**3.6 Dirichlet boundary conditions.** Let us first consider the case where the field fulfills Dirichlet boundary conditions on both the hyperplanes  $\pi_0, \pi_a$ , meaning that

$$\widehat{\phi}(t, \mathbf{x}) = 0 \quad \text{for } t \in \mathbf{R}, \mathbf{x} \in \pi_0 \text{ or } \mathbf{x} \in \pi_a. \quad (3.19)$$

Recall that in this case the cylinder kernel associated to the reduced problem is (see Eq. (6.20) in Part I)

$$T^{(1)}(\mathbf{t}; x^1, y^1) = \frac{1}{2a} \left[ \frac{\cos(\frac{\pi}{a}(x^1 - y^1)) - e^{-\frac{\pi}{a}\mathbf{t}}}{\cosh(\frac{\pi}{a}\mathbf{t}) - \cos(\frac{\pi}{a}(x^1 - y^1))} - \frac{\cos(\frac{\pi}{a}(x^1 + y^1)) - e^{-\frac{\pi}{a}\mathbf{t}}}{\cosh(\frac{\pi}{a}\mathbf{t}) - \cos(\frac{\pi}{a}(x^1 + y^1))} \right]. \quad (3.20)$$

To obtain the renormalized stress-energy VEV in any spatial dimension  $d$ , it suffices to substitute Eq. (3.20) into Eq.s (3.12-3.14) and to compute the residues therein. Let us explicitate the final results for  $d = 3$ ; in this case the residues in (3.12-3.14), involving  $T^{(1)}(\mathbf{t}; x^1, y^1)|_{y^1=x^1}$ , can be derived from the  $\mathbf{t} \rightarrow 0$  expansion

$$\begin{aligned} T^{(1)}(\mathbf{t}; x^1, y^1) \Big|_{y^1=x^1} = \\ \frac{1}{\pi \mathbf{t}} + \frac{\pi(3 - \sin^2(\frac{\pi}{a}x^1))}{12a^2 \sin^2(\frac{\pi}{a}x^1)} \mathbf{t} + \frac{\pi^3(15(2 + \cos(\frac{2\pi}{a}x^1)) - \sin^4(\frac{\pi}{a}x^1))}{720a^4 \sin^4(\frac{\pi}{a}x^1)} \mathbf{t}^3 + O(\mathbf{t}^5). \end{aligned} \quad (3.21)$$

Proceeding in a similar manner where the spatial derivatives of  $T^{(1)}$  appear, we obtain this final result for the  $d = 3$  renormalized VEV of the stress-energy tensor:

$$\begin{aligned} \langle 0 | \widehat{T}_{\mu\nu}(\mathbf{x}) | 0 \rangle_{ren} \Big|_{\mu, \nu=0,1,2,3} &= A \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \left( \xi - \frac{1}{6} \right) B(x^1) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ A &= \frac{\pi^2}{1440a^4}, \quad B(x^1) = \frac{\pi^2}{8a^4} \frac{3 - 2\sin^2(\frac{\pi}{a}x^1)}{\sin^4(\frac{\pi}{a}x^1)} \quad \text{for } x^1 \in (0, a). \end{aligned} \quad (3.22)$$

This is a classical result, whose earlier derivations used point splitting methods rather than zeta regularization (see, e.g., the paper by Esposito et al. [16] or the

already cited monographies [3, 22, 25]); in our previous work [17] the expression (3.22) was obtained via zeta regularization, but the required analytic continuations were derived by *ad hoc* considerations, strictly related to the peculiar configuration under analysis. On the contrary, here the analytic continuation arises automatically from the general schemes that we have developed for an arbitrary geometry <sup>(6)</sup>.

Next, let us pass the reduced bulk energy. To this purpose, recall that the cylinder trace associated to the reduced problem under analysis is (see Eq. (6.25) in Part I)

$$T^{(1)}(\mathbf{t}) = \frac{1}{e^{\frac{\pi}{a}\mathbf{t}} - 1} ; \quad (3.23)$$

now, the general relation (3.16) allows us to infer, for  $d = 3$ ,

$$E_1^{ren} = - \frac{\pi^2}{1440a^3} . \quad (3.24)$$

Finally, using either definition (3.17) or (3.18), we obtain for the non-vanishing component of the pressure on the planes  $\pi_0$  and  $\pi_a$  the following expressions, respectively:

$$p_1^{ren}(\mathbf{x}) \Big|_{\pi_0} = 3A , \quad p_1^{ren}(\mathbf{x}) \Big|_{\pi_a} = -3A \quad (3.25)$$

where  $A$  as in Eq. (3.22). This means that in the present case, the forces on the boundary planes produced by the field in the interior region are attractive <sup>(7)</sup>.

**3.7 Dirichlet-Neumann boundary conditions.** Consider now the parallel hyperplanes configuration where the field fulfills Dirichlet and Neumann boundary conditions, respectively, on the hyperplanes  $\pi_0$  and  $\pi_a$ ; explicitly,

$$\widehat{\phi}(t, \mathbf{x}) = 0 \text{ for } (t, \mathbf{x}) \in \mathbf{R} \times \pi_0 , \quad \partial_{x^1} \widehat{\phi}(t, \mathbf{x}) = 0 \text{ for } (t, \mathbf{x}) \in \mathbf{R} \times \pi_a . \quad (3.26)$$

The cylinder kernel associated to the reduced operator  $\mathcal{A}_1$  is (see Eq. (6.30) in Part I)

$$T^{(1)}(\mathbf{t}; x^1, y^1) = \frac{1}{a} \left[ \frac{\sinh(\frac{\pi}{2a}\mathbf{t}) \cos(\frac{\pi}{2a}(x^1 - y^1))}{\cosh(\frac{\pi}{a}\mathbf{t}) - \cos(\frac{\pi}{a}(x^1 - y^1))} - \frac{\sinh(\frac{\pi}{2a}\mathbf{t}) \cos(\frac{\pi}{2a}(x^1 + y^1))}{\cosh(\frac{\pi}{a}\mathbf{t}) - \cos(\frac{\pi}{a}(x^1 + y^1))} \right] . \quad (3.27)$$

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<sup>(6)</sup>In connection with the present approach, see also [18].

<sup>(7)</sup> Clearly, we are not taking into account any effect related to the outer region (see the comments at the end of subsection 2.12 of Part I). As a matter of fact, the forces produced by a massless scalar field in this region, fulfilling either Dirichlet or Neumann boundary conditions on the planes, vanish identically for  $d = 3$ ; this result can be derived using the methods that will be developed in the following Section 4.

Employing this kernel along with relations (3.12-3.14), we can evaluate the renormalized VEV of the stress-energy tensor; in particular, for  $d = 3$  we obtain

$$\langle 0 | \widehat{T}_{\mu\nu}(\mathbf{x}) | 0 \rangle_{ren} \Big|_{\mu,\nu=0,1,2,3} = A \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} - \left( \xi - \frac{1}{6} \right) B(x^1) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A = \frac{7\pi^2}{11520a^4}, \quad B(x^1) = \frac{\pi^2}{64a^4} \frac{23 \cos(\frac{\pi}{a}x^1) + \cos(\frac{3\pi}{a}x^1)}{\sin^4(\frac{\pi}{a}x^1)} \quad \text{for } x^1 \in (0, a). \quad (3.28)$$

To the best of our knowledge, the full stress-energy VEV for the present configuration has never been given in the preceeding literature; only the global results on energy and pressure discussed hereafter have previously been considered (see the citations below).

Let us recall that the reduced cylinder trace in this case is (see Eq. (6.33) in Part I)

$$T^{(1)}(\mathbf{t}) = \frac{e^{\frac{\pi}{2a}\mathbf{t}}}{e^{\frac{\pi}{a}\mathbf{t}} - 1}. \quad (3.29)$$

The above expression, along with prescription (3.16), allows us to determine the renormalized, reduced bulk energy; for example, for  $d = 3$  we obtain

$$E_1^{ren} = \frac{7\pi^2}{11520a^3}. \quad (3.30)$$

Concerning the non-vanishing component of the boundary pressure, both definitions (3.17) (3.18) give (with  $A$  as in Eq. (3.28))

$$p_1^{ren}(\mathbf{x}) \Big|_{\pi_0} = -3A, \quad p_1^{ren}(\mathbf{x}) \Big|_{\pi_a} = 3A; \quad (3.31)$$

let us stress that, similarly to the results found for the segment configuration in Part I (see Eq.s (6.27) (6.35) therein), the expressions in Eq. (3.31) have the opposite sign with respect to those in Eq. (3.25), holding in the case of Dirichlet conditions on both the planes  $\pi_0$  and  $\pi_a$ . This means that in the present Dirichlet-Neumann case, the forces on the boundary planes are repulsive<sup>(8)</sup>.

The results in Eq.s (3.30) (3.31) agree with those reported, e.g., in [10, 28] and Santos et al. [32] (see also [2, 4]).

**3.8 Neumann boundary conditions.** Assume that

$$\partial_{x^1} \widehat{\phi}(t, \mathbf{x}) = 0 \quad \text{for } t \in \mathbf{R}, \text{ and } \mathbf{x} \in \pi_0 \text{ or } \mathbf{x} \in \pi_a. \quad (3.32)$$

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<sup>8</sup>Recall the considerations of the previous footnote 7.



Recall that in this case, according to the considerations of subsection 2.13, the reduced operator  $\mathcal{A}_1$  must be viewed as acting in the Hilbert space  $L_0^2(0, a)$  of square integrable functions on  $(0, a)$  with mean zero (see Eq. (2.60)). The cylinder kernel of  $\mathcal{A}_1$  is (see Eq. (6.38) in Part I)

$$T^{(1)}(\mathbf{t}; x^1, y^1) = \frac{1}{2a} \left[ \frac{\cos(\frac{\pi}{a}(\mathbf{x}-\mathbf{y})) - e^{-\mathbf{t}}}{\cosh \mathbf{t} - \cos(\frac{\pi}{a}(\mathbf{x}-\mathbf{y}))} + \frac{\cos(\frac{\pi}{a}(\mathbf{x}+\mathbf{y})) - e^{-\mathbf{t}}}{\cosh \mathbf{t} - \cos(\frac{\pi}{a}(\mathbf{x}+\mathbf{y}))} \right]. \quad (3.33)$$

Using this kernel along with relations (3.12-3.14), we obtain the renormalized stress-energy VEV; in particular, for  $d = 3$  the residue evaluation yields

$$\begin{aligned} & \langle 0 | \widehat{T}_{\mu\nu}(\mathbf{x}) | 0 \rangle_{ren} \Big|_{\mu, \nu=0,1,2,3} = \\ & = A \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \left( \xi - \frac{1}{6} \right) B(x^1) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (3.34)$$

where  $A$  and  $B(x^1)$  are defined as in Eq. (3.22).

Let us pass to the reduced bulk energy; in subsection 6.8 of Part I we noticed that in this case the cylinder trace  $T^{(1)}(\mathbf{t})$  coincides with the one of Eq. (3.23), corresponding to the case of Dirichlet boundary conditions. In consequence of this the renormalized, reduced bulk energy for  $d = 3$  is the same as in the case of Dirichlet boundary conditions (see Eq. (3.24)):

$$E_1^{ren} = - \frac{\pi^2}{1440a^3}.$$

Furthermore, concerning the pressure on the boundary, both definitions (3.17) and (3.18) also give the same result we derived in the case of Dirichlet boundary conditions (see Eq. (3.25); again,  $A$  is the coefficient of Eq. (3.22))

$$p_1^{ren}(\mathbf{x}) \Big|_{\pi_0} = 3A, \quad p_1^{ren}(\mathbf{x}) \Big|_{\pi_a} = -3A.$$

**3.9 Periodic boundary conditions.** The last case we consider is the one where

$$\begin{aligned} \widehat{\phi}(t, 0, x^2, \dots, x^d) &= \widehat{\phi}(t, a, x^2, \dots, x^d), & \partial_{x^1} \widehat{\phi}(t, 0, x^2, \dots, x^d) &= \partial_{x^1} \widehat{\phi}(t, a, x^2, \dots, x^d) \\ & \text{for } t, x^2, \dots, x^d \in \mathbf{R}. \end{aligned} \quad (3.35)$$

Similarly to what was said for the segment with periodic boundary conditions, note that, as explained in subsection 2.15, this configuration would be more properly formulated in terms of a free scalar field on the flat manifold  $\Omega := \mathbf{T}_a^1 \times \mathbf{R}^{d-1}$ , where the first factor is the torus  $\mathbf{T}_a^1 := \mathbf{R}/(a\mathbf{Z})$ .

As in the case of Neumann boundary conditions, the basic Hilbert space for the reduced problem is  $L_0^2(\mathbf{T}_a^1)$  (see subsection 2.13, Eq. (2.60)). We know that the cylinder kernel associated to  $\mathcal{A}_1$  in this case is (see Eq. (6.44) in Part I)

$$T^{(1)}(\mathbf{t}; x^1, y^1) = \frac{\cos(\frac{2\pi}{a}(x^1 - y^1)) - e^{-\frac{2\pi}{a}\mathbf{t}}}{a [\cosh(\frac{2\pi}{a}\mathbf{t}) - \cos(\frac{2\pi}{a}(x^1 - y^1))]} . \quad (3.36)$$

Again, we can employ Eq.s (3.12-3.14) to evaluate the renormalized stress-energy VEV. Differently from the previous subcases, this time the expressions appearing in intermediate steps of the required calculations are simple enough to be reported; as an example, let us focus on the evaluation of the component  $\langle 0 | \widehat{T}_{00}(\mathbf{x}) | 0 \rangle_{ren}$ . First of all, note that

$$T^{(1)}(\mathbf{t}; x^1, x^1) = \frac{1}{a} \left[ \coth\left(\frac{\pi}{a}\mathbf{t}\right) - 1 \right] ; \quad (3.37)$$

$$\partial_{x^1 y^1} T^{(1)}(\mathbf{t}; x^1, x^1) = \frac{2\pi^2}{a^3} \coth\left(\frac{\pi}{a}\mathbf{t}\right) \operatorname{csch}^2\left(\frac{\pi}{a}\mathbf{t}\right) . \quad (3.38)$$

So, after some simple algebraic manipulations, Eq. (3.12) yields

$$\begin{aligned} \langle 0 | \widehat{T}_{00}(\mathbf{x}) | 0 \rangle_{ren} = & -\frac{C_d}{4a} \operatorname{Res} \left( \frac{2 - (1 - 4\xi)d}{\mathbf{t}^{d+1}} \left[ \coth\left(\frac{\pi}{a}\mathbf{t}\right) - 1 \right] + \right. \\ & \left. + \frac{2(1 - 4\xi)}{(d-1)\mathbf{t}^{d+1}} \left(\frac{\pi}{a}\mathbf{t}\right)^2 \coth\left(\frac{\pi}{a}\mathbf{t}\right) \operatorname{csch}^2\left(\frac{\pi}{a}\mathbf{t}\right); 0 \right) . \end{aligned} \quad (3.39)$$

The function in the above expression, whose residue in  $\mathbf{t} = 0$  is required, is easily seen to be meromorphic with a pole of order  $d + 2$  in  $\mathbf{t} = 0$ ; more precisely, its Laurent expansion is

$$\begin{aligned} & -\frac{d(d-3) + 4(d^2 - d - 2)\xi}{4\pi(d-1)} \frac{1}{\mathbf{t}^{d+2}} + \frac{(2-d) + 4d\xi}{4a} \frac{1}{\mathbf{t}^{d+1}} + \\ & + \frac{\pi((2-d) + 4d\xi)}{12a^2} \frac{1}{\mathbf{t}^d} - \frac{\pi^3(-(d^2 - 3d - 4) + 4(d^2 - d - 6)\xi)}{180(d-1)a^4} \frac{1}{\mathbf{t}^{d-2}} + O(\mathbf{t}^{4-d}) . \end{aligned} \quad (3.40)$$

For example, for  $d = 3$  Eq. (3.39) yields ( $A_3 = -1/\pi$ , see Eq. (3.10))

$$\langle 0 | \widehat{T}_{00}(\mathbf{x}) | 0 \rangle_{ren} = -\frac{\pi^2}{90a^4} ; \quad (3.41)$$

proceeding similarly for the other components of the renormalized stress-energy VEV (for  $d = 3$ ), we obtain

$$\langle 0 | \widehat{T}_{\mu\nu}(\mathbf{x}) | 0 \rangle_{ren} \Big|_{\mu, \nu=0,1,2,3} = \frac{\pi^2}{90a^4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \quad (3.42)$$

Notice that the renormalized stress-energy tensor (3.42) does not depend explicitly on the periodic coordinate  $x^1$ . This comes as no surprise; indeed, it reflects the invariance of the theory under translations  $x^1 \mapsto x^1 + \alpha$  (for arbitrary  $\alpha \in \mathbf{R}$ ). To conclude, let us consider the total bulk energy. First, recall that the reduced cylinder trace in this case is (see Eq. (6.47) in Part I)

$$T(\mathbf{t}) = \frac{2}{e^{\frac{2\pi}{a}\mathbf{t}} - 1} ; \quad (3.43)$$

so, using once more the prescription (3.16) we obtain

$$E^{ren} = -\frac{\pi^2}{90a^3} . \quad (3.44)$$

In conclusion, let us stress that the above expression for the total energy agrees with the results in [4, 14, 15, 25].

## 4 The case of massive field constrained by perpendicular hyperplanes

**4.1 Introducing the problem.** We consider a scalar field of nonzero mass  $m$  fulfilling either Dirichlet or Neumann boundary conditions on  $d_1$  orthogonal hyperplanes in  $d = d_1 + d_2$  spatial dimension. More precisely, we are interested in the case

$$\Omega = (0, +\infty)^{d_1} \times \mathbf{R}^{d_2} , \quad V(\mathbf{x}) = m^2 \quad (m > 0) . \quad (4.1)$$

The domain  $\Omega$  is bounded by the hyperplanes  $\{x^1 = 0\}, \dots, \{x^{d_1} = 0\}$ ; its boundary is the union of the faces

$$\pi_n := \{\mathbf{x} \in \partial\Omega \mid x^n = 0\} , \quad (n \in \{1, \dots, d_1\}) \quad (4.2)$$

and, for each one of them, either Dirichlet or Neumann boundary conditions are prescribed. As anticipated in the Introduction, the same configuration with at most three faces was considered by Actor et al. in [1, 2]; when possible we will establish connections with these works, making direct comparison.

Before proceeding, let us stress that also in this case we can use the results of subsection 2.10 on slab configurations; because of this, we will consider the reduced problem based on

$$\Omega_1 = (0, +\infty)^{d_1} , \quad \mathcal{A}_1 := -\Delta_1 + m^2 , \quad (4.3)$$

with the appropriate boundary conditions.

**4.2 The reduced heat kernel.** In our approach, a basic step for the analysis of the reduced problem (4.3) is the computation of the heat kernel  $K^{(1)}(\mathbf{t}; \mathbf{x}_1, \mathbf{y}_1)$  associated to  $\mathcal{A}_1$ . The result is <sup>(9)</sup>

$$K^{(1)}(\mathbf{t}; \mathbf{x}_1, \mathbf{y}_1) = \frac{e^{-m^2 \mathbf{t}}}{(4\pi \mathbf{t})^{d_1/2}} \prod_{n=1}^{d_1} \left( e^{-\frac{(x_1^n - y_1^n)^2}{4\mathbf{t}}} + \alpha_n e^{-\frac{(x_1^n + y_1^n)^2}{4\mathbf{t}}} \right), \quad (4.4)$$

where, for any  $n \in \{1, \dots, d_1\}$ ,  $\alpha_n \in \mathbf{R}$  is a parameter distinguishing between Dirichlet and Neumann boundary conditions on the face  $\pi_n$ ; more precisely,

$$\alpha_n := \begin{cases} -1 & \text{for Dirichlet B.C. on } \pi_n \\ +1 & \text{for Neumann B.C. on } \pi_n \end{cases} \quad (n \in \{1, \dots, d_1\}). \quad (4.5)$$

Eq. (4.4) can be re-written as <sup>(10)</sup>

$$K^{(1)}(\mathbf{t}; \mathbf{x}_1, \mathbf{y}_1) = \frac{e^{-m^2 \mathbf{t}}}{(4\pi \mathbf{t})^{d_1/2}} \sum_{n=0}^{d_1} \frac{1}{(d_1 - n)! n!} \sum_{\sigma \in S_{d_1}} \alpha_{n, \sigma} e^{-\frac{1}{4\mathbf{t}} \left( \sum_{i=1}^n (x_1^{\sigma(i)} - y_1^{\sigma(i)})^2 + \sum_{j=n+1}^{d_1} (x_1^{\sigma(j)} + y_1^{\sigma(j)})^2 \right)}. \quad (4.6)$$

---

<sup>9</sup>Here is one way to derive Eq. (4.4). First, notice that a complete orthonormal system of (improper) eigenfunctions of  $\mathcal{A}_1 = -\Delta_1 + m^2$  on  $\Omega_1 = (0, +\infty)^{d_1}$  fulfilling the prescribed (either Dirichlet or Neumann) boundary conditions on  $\partial\Omega_1$  is given by

$$F_{\mathbf{k}}(\mathbf{x}_1) := \frac{1}{(2\pi)^{d_1/2}} \prod_{n=1}^{d_1} \left( e^{ik_n x_1^n} + \alpha_n e^{-ik_n x_1^n} \right), \quad \omega_{\mathbf{k}} := \sqrt{|\mathbf{k}|^2 + m^2} \quad \text{for } \mathbf{k} \in \mathcal{K} \equiv (0, +\infty)^{d_1};$$

here, for any  $n \in \{1, \dots, d_1\}$ ,  $\alpha_n$  is defined according to Eq. (4.5). Then, using the eigenfunction expansion (2.26) of the heat kernel, we obtain

$$\begin{aligned} K^{(1)}(\mathbf{t}; \mathbf{x}_1, \mathbf{y}_1) &= \frac{e^{-m^2 \mathbf{t}}}{(2\pi)^{d_1}} \prod_{n=1}^{d_1} \int_0^{+\infty} dk e^{-\mathbf{t} k^2} \left( e^{ik x_1^n} + \alpha_n e^{-ik x_1^n} \right) \left( e^{-ik y_1^n} + \alpha_n e^{ik y_1^n} \right) = \\ &= \frac{e^{-m^2 \mathbf{t}}}{(2\pi)^{d_1}} \prod_{n=1}^{d_1} \int_{-\infty}^{+\infty} dk e^{-\mathbf{t} k^2} \left( e^{ik(x_1^n - y_1^n)} + \alpha_n e^{ik(x_1^n + y_1^n)} \right); \end{aligned}$$

Eq. (4.4) follows by explicitly evaluating every single Gaussian integral in the product.

<sup>10</sup> This result depends on the identity

$$\prod_{n=1}^d (a_n + b_n) = \sum_{n=0}^d \frac{1}{(d-n)! n!} \sum_{\sigma \in S_d} \left( \prod_{i=1}^n a_{\sigma(i)} \right) \left( \prod_{j=n+1}^d b_{\sigma(j)} \right)$$

holding for any  $d \in \{1, 2, 3, \dots\}$ ,  $a_n, b_n \in \mathbf{R}$  ( $n \in \{1, 2, \dots, d\}$ ), where by convention we intend  $\prod_{i=1}^0 a_{\sigma(i)} := \prod_{j=d+1}^d b_{\sigma(j)} := 1$ .

Here and in the following  $S_{d_1}$  denotes the symmetric group with  $d_1$  elements and, by convention, the sums over  $i$  and  $j$  in (4.6) are zero for  $n = 0$  and  $n = d_1$ , respectively; moreover, for any  $\sigma \in S_{d_1}$ , we put

$$\alpha_{n,\sigma} := \prod_{l=n+1}^{d_1} \alpha_{\sigma(l)} \quad \text{for } n \in \{0, \dots, d_1-1\}, \quad \alpha_{d_1,\sigma} := 1. \quad (4.7)$$

**4.3 The reduced Dirichlet kernel.** In the following we will use Eq. (2.30) to express the Dirichlet kernel  $D_s^{(1)}(\mathbf{x}_1, \mathbf{y}_1)$  associated to  $\mathcal{A}_1$ , along with its analytic continuation, in terms of  $K^{(1)}$ . Substituting the expression (4.6) for the heat kernel into Eq. (2.30), we obtain

$$D_s^{(1)}(\mathbf{x}_1, \mathbf{y}_1) = \frac{1}{(4\pi)^{d_1/2} \Gamma(s)} \sum_{n=0}^{d_1} \frac{1}{(d_1-n)!n!} \sum_{\sigma \in S_{d_1}} \alpha_{n,\sigma} \mathfrak{J}_{s,n,\sigma}(\mathbf{x}_1, \mathbf{y}_1), \quad (4.8)$$

where, for each  $n \in \{0, \dots, d_1\}$  and each  $\sigma \in S_{d_1}$ ,

$$\mathfrak{J}_{s,n,\sigma}(\mathbf{x}_1, \mathbf{y}_1) := \int_0^{+\infty} dt \, t^{s-\frac{d_1}{2}-1} e^{-m^2 t - \frac{1}{4t} \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)}, \quad (4.9)$$

$$\mathfrak{N}_{n,\sigma}(\mathbf{x}_1, \mathbf{y}_1) := \left( \sum_{i=1}^n (x_1^{\sigma(i)} - y_1^{\sigma(i)})^2 + \sum_{j=n+1}^{d_1} (x_1^{\sigma(j)} + y_1^{\sigma(j)})^2 \right)^{1/2}. \quad (4.10)$$

Clearly  $\mathfrak{N}_{n,\sigma}(\mathbf{x}_1, \mathbf{y}_1) \geq 0$ , so that convergence conditions for the integral in Eq. (4.9) can be readily inferred; more precisely, if  $\mathfrak{N}_{n,\sigma}(\mathbf{x}_1, \mathbf{y}_1) > 0$  the integral converges for any  $s \in \mathbf{C}$  while if  $\mathfrak{N}_{n,\sigma}(\mathbf{x}_1, \mathbf{y}_1) = 0$  (which, in the interior of  $\Omega$ , happens if and only if  $n = d_1$  and  $\mathbf{y}_1 = \mathbf{x}_1$ ) it only converges for

$$\Re s > \frac{d_1}{2}. \quad (4.11)$$

Under these assumptions for convergence, Eq. (4.9) is strictly related to a known integral representation of the modified Bessel function of the second kind  $K_\nu$  (see, e.g., [27], pag.253, Eq.10.32.10); using this representation we obtain, for any  $\mathbf{x}_1, \mathbf{y}_1 \in (0, +\infty)^{d_1}$ ,

$$\mathfrak{J}_{s,n,\sigma}(\mathbf{x}_1, \mathbf{y}_1) = 2^{\frac{d_1}{2}+1-s} m^{d_1-2s} \mathfrak{G}_{s-\frac{d_1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)), \quad (4.12)$$

where, for the sake of brevity, we have put

$$\mathfrak{G}_\nu : [0, +\infty) \rightarrow \mathbf{C}, \quad u \mapsto \mathfrak{G}_\nu(z) := z^{\nu/2} K_\nu(\sqrt{z}). \quad (4.13)$$

In conclusion

$$D_s^{(1)}(\mathbf{x}_1, \mathbf{y}_1) = \frac{2^{1-s} m^{d_1-2s}}{(2\pi)^{d_1/2} \Gamma(s)} \sum_{n=0}^{d_1} \frac{1}{(d_1-n)!n!} \sum_{\sigma \in S_{d_1}} \alpha_{n,\sigma} \mathfrak{G}_{s-\frac{d_1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) . \quad (4.14)$$

The derivatives of  $D_s^{(1)}$  of any order can be computed using the identity <sup>(11)</sup>

$$\frac{d^n \mathfrak{G}_\nu}{dz^n}(z) = \left(-\frac{1}{2}\right)^n \mathfrak{G}_{\nu-n}(z) \quad \text{for } n \in \{1, 2, 3, \dots\} ; \quad (4.15)$$

moreover, in the cases where  $\mathfrak{N}_{n,\sigma}(\mathbf{x}_1, \mathbf{y}_1) = 0$ , we can resort to the relation

$$\mathfrak{G}_\nu(0) = 2^{\nu-1} \Gamma(\nu) \quad \text{for } \nu \in \mathbf{C}, \Re \nu > 0 . \quad (4.16)$$

**4.4 The  $d$ -dimensional Dirichlet kernel.** Using the previous results and Eq.s (2.46-2.49), we obtain the following expressions for the Dirichlet kernel  $D_s(\mathbf{x}, \mathbf{y})$  of the  $d$ -dimensional problem (4.1) and for its derivatives, along the diagonal  $\mathbf{y} = \mathbf{x}$ :

$$D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \frac{2^{\frac{2\mp 1-u}{2}} m^{d\mp 1-u}}{(2\pi)^{d/2} \Gamma(\frac{u\pm 1}{2})} \sum_{n=0}^{d_1} \frac{1}{(d_1-n)!n!} \cdot \sum_{\sigma \in S_{d_1}} \alpha_{n,\sigma} \mathfrak{G}_{\frac{u-d\pm 1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) \Big|_{\mathbf{y}_1=\mathbf{x}_1} ; \quad (4.17)$$

$$\partial_{z_1^i w_1^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \frac{2^{\frac{1-u}{2}} m^{d-1-u}}{(2\pi)^{d/2} \Gamma(\frac{u+1}{2})} \sum_{n=0}^{d_1} \frac{1}{(d_1-n)!n!} \cdot \sum_{\sigma \in S_{d_1}} \alpha_{n,\sigma} \partial_{z_1^i w_1^j} \mathfrak{G}_{\frac{u-d+1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) \Big|_{\mathbf{y}_1=\mathbf{x}_1} \quad (4.18)$$

for  $z, w \in \{x, y\}$  and  $i, j \in \{1, \dots, d_1\}$  ;

$$\begin{aligned} \partial_{x_2^i y_2^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} &= -\partial_{x_2^i x_2^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = -\partial_{y_2^i y_2^j} D_{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \\ &= \delta_{ij} \frac{2^{\frac{1-u}{2}} m^{d+1-u}}{(2\pi)^{d/2} \Gamma(\frac{u+1}{2})} \sum_{n=0}^{d_1} \frac{1}{(d_1-n)!n!} \sum_{\sigma \in S_{d_1}} \alpha_{n,\sigma} \mathfrak{G}_{\frac{u-d-1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) \Big|_{\mathbf{y}_1=\mathbf{x}_1} \end{aligned} \quad (4.19)$$

for  $i, j \in \{1, \dots, d_2\}$  .

Note that, in each of the sums over  $n$  appearing in the above expressions, the terms corresponding to  $n \in \{0, 1, \dots, d_1 - 1\}$  are analytic functions of  $u$  on the whole

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<sup>11</sup>We already considered the map  $\mathfrak{G}_\nu$  of Eq. (4.13) in Appendix D of Part I; therein we also showed how to derive the relations (4.15) (4.16).

complex plane since  $\mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{x}_1) > 0$  for any  $\mathbf{x}_1 \in (0, +\infty)^{d_1}$ . On the contrary, the terms corresponding to  $n = d_1$  deserve particular attention; indeed,  $\mathfrak{N}_{d_1,\sigma}^2(\mathbf{x}_1, \mathbf{x}_1) = 0$  for any  $\mathbf{x}_1 \in (0, +\infty)^{d_1}$  so that, in order to evaluate these contributions, we have to resort to Eq. (4.16) (also recalling Eq. (4.15)). In this way we obtain

$$\mathfrak{G}_{\frac{u-d+1}{2}}(m^2 \mathfrak{N}_{d_1,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) \Big|_{\mathbf{y}_1=\mathbf{x}_1} = 2^{\frac{u-d+1}{2}-1} \Gamma\left(\frac{u-d+1}{2}\right); \quad (4.20)$$

$$\partial_{z_1^i w_1^j} \mathfrak{G}_{\frac{u-d+1}{2}}(m^2 \mathfrak{N}_{d_1,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) \Big|_{\mathbf{y}_1=\mathbf{x}_1} = -\delta_{ij} (2\delta_{zw} - 1) m^2 2^{\frac{u-d-3}{2}} \Gamma\left(\frac{u-d-1}{2}\right) \quad (4.21)$$

for  $z, w \in \{x, y\}$  and  $i, j \in \{1, \dots, d_1\}$ .

In principle, the above equations hold with the limitations on  $u$  arising from Eq. (4.16); more precisely, Eq. (4.20) holds for  $\Re u > d-1$  and Eq. (4.21) for  $\Re u > d+1$ . However, the right-hand sides of these equations are well defined and analytic on the whole complex plane with the exception of simple poles placed at

$$u = d + 1 - 2\ell \quad \text{for } \ell \in \{0, 1, 2, \dots\}; \quad (4.22)$$

this remark gives the meromorphic continuation in  $u$  of the functions in Eq.s (4.20) (4.21) and, consequently, of the terms with  $n = d_1$  in Eq.s (4.17-4.19).

**4.5 The stress-energy tensor.** Using Eq.s (2.16-2.18) along with Eq.s (4.17-4.19) (and Eq.s (4.20-4.21) for the terms with  $n = d_1$ ), we obtain the analytic continuation to a meromorphic function of each component of the regularized stress-energy VEV, required in order to implement the local zeta approach. In particular, we have the following expressions for the non-vanishing components:

$$\begin{aligned} \langle 0 | \hat{T}_{00}^u(\mathbf{x}) | 0 \rangle &= \frac{2^{\frac{1-u}{2}} m^{d+1}}{(2\pi)^{d/2} \Gamma(\frac{u+1}{2})} \left(\frac{m}{\kappa}\right)^{-u} \sum_{n=0}^{d_1} \frac{1}{(d_1-n)! n!} \sum_{\sigma \in S_{d_1}} \alpha_{n,\sigma} \cdot \\ &\cdot \left[ \left( \frac{d-d_1-1+u}{4} - (d-d_1+1-u)\xi \right) \mathfrak{G}_{\frac{u-d-1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) + \right. \\ &\quad \left. + \left( \frac{1}{4} - \xi \right) (1 + m^{-2} \partial_{x_1}^\ell \partial_{y_1}^\ell) \mathfrak{G}_{\frac{u-d+1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) \right] \Big|_{\mathbf{y}_1=\mathbf{x}_1}; \end{aligned} \quad (4.23)$$

$$\begin{aligned}
\langle 0 | \widehat{T}_{ij}^u(\mathbf{x}) | 0 \rangle &= \langle 0 | \widehat{T}_{ji}^u(\mathbf{x}) | 0 \rangle = \frac{2^{\frac{1-u}{2}} m^{d+1}}{(2\pi)^{d/2} \Gamma(\frac{u+1}{2})} \left(\frac{m}{\kappa}\right)^{-u} \sum_{n=0}^{d_1} \frac{1}{(d_1-n)! n!} \sum_{\sigma \in S_{d_1}} \alpha_{n,\sigma} \cdot \\
&\cdot \left[ -\left(\frac{1}{4} - \xi\right) \delta_{ij} \left( (d-d_1+1-u) \mathfrak{G}_{\frac{u-d-1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) + \mathfrak{G}_{\frac{u-d+1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) \right) + \right. \\
&\left. + m^{-2} \left( -\left(\frac{1}{4} - \xi\right) \delta_{ij} \partial_{y_1^\ell}^{x_1^\ell} + \left(\frac{1}{2} - \xi\right) \partial_{x_1^i y_1^j} - \xi \partial_{x_1^i x_1^j} \right) \mathfrak{G}_{\frac{u-d+1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) \right]_{\mathbf{y}_1 = \mathbf{x}_1} \\
&\text{for } i, j \in \{1, \dots, d_1\} ; \tag{4.24}
\end{aligned}$$

$$\begin{aligned}
\langle 0 | \widehat{T}_{ij}^u(\mathbf{x}) | 0 \rangle &= \langle 0 | \widehat{T}_{ji}^u(\mathbf{x}) | 0 \rangle = -\delta_{ij} \frac{2^{\frac{1-u}{2}} m^{d+1}}{(2\pi)^{d/2} \Gamma(\frac{u+1}{2})} \left(\frac{m}{\kappa}\right)^{-u} \sum_{n=0}^{d_1} \frac{1}{(d_1-n)! n!} \sum_{\sigma \in S_{d_1}} \alpha_{n,\sigma} \cdot \\
&\cdot \left[ \left( \frac{d-d_1-1-u}{4} - (d-d_1+1-u)\xi \right) \mathfrak{G}_{\frac{u-d-1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) + \right. \\
&\left. + \left(\frac{1}{4} - \xi\right) (1 + m^{-2} \partial_{y_1^\ell}^{x_1^\ell}) \mathfrak{G}_{\frac{u-d+1}{2}}(m^2 \mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{y}_1)) \right]_{\mathbf{y}_1 = \mathbf{x}_1} \\
&\text{for } i, j \in \{d_1+1, \dots, d_1+d_2 \equiv d\} . \tag{4.25}
\end{aligned}$$

The renormalized VEV of the stress-energy tensor is obtained sending  $u$  to zero in the above expressions; the only singularities appear in the terms corresponding to  $n = d_1$ , which must be treated resorting to Eq.s (4.20-4.22).

The conclusions are the following:

- i) For  $d$  even each component of the regularized stress-energy VEV is an analytic function of  $u$  near  $u = 0$ ; thus its renormalized version is obtained via the restricted zeta approach, i.e., by simply evaluating Eq.s (4.23-4.25) at  $u = 0$ .
- ii) For  $d$  odd the regularized stress-energy VEV has a simple pole in  $u = 0$ , so that we have to resort to the extended zeta approach and consider the regular part at  $u = 0$ .

The manipulations indicated in i) are trivial; the ones indicated in ii) could be performed in principle for an arbitrary odd dimension, but the final expressions are too lengthy to be reported here. For this reason, we prefer to exemplify ii) in two special cases with  $d = 3$  (see subsections 4.8 and 4.9).

**4.6 The boundary forces.** As in the previous section, following the general framework of subsection 2.12, we can give two alternative definitions for the pressure acting on the boundary of the spatial domain  $\Omega$ .

Let us consider a point  $\mathbf{x} \in \partial\Omega$ ; if  $d_1 > 1$  we exclude  $\mathbf{x}$  to be on a corner, where the outer normal is ill-defined. To fix our ideas, we assume that  $\mathbf{x}$  is an inner point of



the face

$$\pi_1 := \{x^1 = 0\} \cap \partial\Omega ; \quad (4.26)$$

let  $\mathbf{n}(\mathbf{x})$  denote the unit outer normal at  $\mathbf{x}$ , so that  $\mathbf{n}(\mathbf{x}) = (-1, 0, \dots, 0)$ . On the one hand, we can define

$$p_i^{ren}(\mathbf{x}) := RP \Big|_{u=0} \langle 0 | \widehat{T}_{ij}^u(\mathbf{x}) | 0 \rangle n^j(\mathbf{x}) = -RP \Big|_{u=0} \langle 0 | \widehat{T}_{i1}^u(\mathbf{x}) | 0 \rangle . \quad (4.27)$$

On the other hand, we can consider the alternative definition

$$p_i^{ren}(\mathbf{x}) := \left( \lim_{\mathbf{x}' \in \Omega, \mathbf{x}' \rightarrow \mathbf{x}} \langle 0 | \widehat{T}_{ij}(\mathbf{x}') | 0 \rangle_{ren} \right) n^j(\mathbf{x}) = - \left( \lim_{\mathbf{x}' \in \Omega, \mathbf{x}' \rightarrow \mathbf{x}} \langle 0 | \widehat{T}_{i1}(\mathbf{x}') | 0 \rangle_{ren} \right) . \quad (4.28)$$

As a matter of fact, *the alternatives (4.27) (4.28) give the same result for the renormalized pressure*; the rest of the present subsection is mainly devoted to the justification of this statement, which requires a nontrivial analysis.

Consider the expression (4.24) of  $\langle 0 | \widehat{T}_{i1}^u | 0 \rangle$ . Due to the considerations in the previous subsections, it is apparent that the terms in (4.24) deserving special attention when comparing the definitions (4.27) (4.28) for the pressure at  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \pi_1$  are those with  $n, \sigma$  such that

$$\mathfrak{N}_{n,\sigma}^2(\mathbf{x}_1, \mathbf{x}_1) = 0 \quad \text{and} \quad \mathfrak{N}_{n,\sigma}^2(\mathbf{x}', \mathbf{x}') \neq 0 \text{ for } \mathbf{x}' \equiv (\mathbf{x}'_1, \mathbf{x}'_2) \in \Omega ; \quad (4.29)$$

these are easily seen to correspond to the choices

$$n = d_1 - 1 \text{ and } \sigma \in S_{d_1} \text{ such that } \sigma(d_1) = 1 . \quad (4.30)$$

Indeed, all the terms corresponding to values of  $n, \sigma$  different from the above ones are straightforwardly seen to yield the same results according to both the prescriptions (4.27) and (4.28). Now, let us focus on the potentially troublesome terms described in Eq. (4.24), corresponding to a choice of the form (4.30); in the sequel we will show that these terms *do not contribute to*  $p_i^{ren}(\mathbf{x})$  for both the alternatives (4.27) (4.28). In order to prove this, let us denote with  $\mathbf{x}' = (\mathbf{x}'_1, \mathbf{x}'_2)$  either the previously mentioned boundary point  $\mathbf{x} \in \pi_1$  or a point of  $\Omega$ . In the expression (4.24) for  $\langle 0 | \widehat{T}_{i1}^u(\mathbf{x}') | 0 \rangle$  we pick up any problematic term with  $n, \sigma$  as in Eq. (4.30); this reads

$$\begin{aligned} & \left[ - \left( \frac{1}{4} - \xi \right) \delta_{i1} \left( (d - d_1 + 1 - u) \mathfrak{G}_{\frac{u-d-1}{2}}(m^2 \mathfrak{N}_{d_1-1,\sigma}^2) + \mathfrak{G}_{\frac{u-d+1}{2}}(m^2 \mathfrak{N}_{d_1-1,\sigma}^2) \right) + \right. \\ & \left. + m^{-2} \left( - \left( \frac{1}{4} - \xi \right) \delta_{i1} \partial^{x'^\ell_1} \partial_{y'^\ell_1} + \left( \frac{1}{2} - \xi \right) \partial_{x'^i_1 y'^1_1} - \xi \partial_{x'^i_1 x'^1_1} \right) \mathfrak{G}_{\frac{u-d+1}{2}}(m^2 \mathfrak{N}_{d_1-1,\sigma}^2) \right]_{\mathbf{y}'_1 = \mathbf{x}'_1} \end{aligned} \quad (4.31)$$

(some of the arguments have been suppressed for the sake of brevity). With some effort, noting that  $\mathfrak{N}_{d_1-1,\sigma}^2(\mathbf{x}'_1, \mathbf{y}'_1) = (x'_1 + y'_1)^2 + \sum_{i=2}^{d_2} (x'_1 - y'_1)^2$  (due to  $\sigma(d_1) = 1$ ) and using Eq. (4.15), we can re-write expression (4.31) as

$$- \delta_{i1} \left( \frac{1}{4} - \xi \right) \left[ (d+1-u) \mathfrak{G}_{\frac{u-d-1}{2}}(z^2) + \mathfrak{G}_{\frac{u-d+1}{2}}(z^2) - z^2 \mathfrak{G}_{\frac{u-d-3}{2}}(z^2) \right]_{z=2mx'_1} \quad (4.32)$$

$$\equiv f(u, x'_1) .$$

We now claim that

$$f(u, 0) \Big|_{u=0} = 0 \quad (4.33)$$

and

$$\lim_{x'_1 \rightarrow 0} \left( f(u, x'_1) \Big|_{u=0} \right) = 0 ; \quad (4.34)$$

let us remark that in both the above equations the prescription of taking the regular part is superfluous, and  $|_{u=0}$  indicates the analytic continuation at  $u = 0$ . Eq.s (4.33) (4.34) state that the problematic terms do not contribute to  $p_i^{ren}(\mathbf{x})$  as defined by Eq.s (4.27) and (4.28), respectively.

In order to prove Eq. (4.33) we note that

$$f(u, 0) = -\delta_{i1} \left( \frac{1}{4} - \xi \right) 2^{\frac{u-d-1}{2}} \left[ \frac{d+1-u}{2} \Gamma\left(\frac{u-d-1}{2}\right) + \Gamma\left(\frac{u-d+1}{2}\right) \right] = 0 \quad (4.35)$$

where in the first passage we used Eq. (4.16), while equality to zero follows from the well-known relation  $\Gamma(z+1) = z \Gamma(z)$ .

To prove Eq. (4.34), recalling the definition (4.13) we infer (for all  $x'_1 > 0$ )

$$f(u, x'_1) \Big|_{u=0} =$$

$$= -\delta_{i1} \left( \frac{1}{4} - \xi \right) \left[ z^{-\frac{d+1}{2}} \left( (d+1) K_{\frac{d+1}{2}}(z) + z K_{\frac{d-1}{2}}(z) - z K_{\frac{d+3}{2}}(z) \right) \right]_{z=2mx'_1} = 0 ;$$

in this case equality to zero follows from the identity below, holding for Bessel functions  $K_\nu$  of any order (see [27], p.251, Eq.10.29.1):

$$z K_{\nu+1}(z) - z K_{\nu-1}(z) = 2\nu K_\nu(z) . \quad (4.37)$$

In the above we assumed  $\mathbf{x}$  to belong to the face with  $x^1 = 0$  but, of course, similar considerations also hold for all the other boundary points not on the corners.

Now, let us spend a few words about points on the corners of  $\partial\Omega$ , which appear if  $d_1 > 1$ ; we already noticed that the outer normal is ill-defined at these points, so that the notion of pressure is itself problematic. The natural strategies that could

be guessed to overcome the problem make apparent some pathologies that we prefer to describe in an example, rather than in general: see subsection 4.9.

In passing, let us anticipate that an analysis similar to the one of this subsection will also be given in Part IV (see subsection 3.6 therein), for the case of a massless field confined within a  $d$ -dimensional box and fulfilling Dirichlet boundary conditions. As in the present setting, the alternative definitions (2.57) (2.58) will be found to agree at all boundary points except those on the corners, where pathologies appear.

**4.7 Introducing two examples.** The framework developed in the previous subsections will be illustrated hereafter, for  $d = 3$ , in these cases:  $\Omega := (0, +\infty) \times \mathbf{R}^2$ , representing a half-space, and  $\Omega := (0, +\infty)^2 \times \mathbf{R}$ , representing a wedge bounded by orthogonal half-planes. In both cases, we consider Dirichlet and/or Neumann boundary conditions.

**4.8 A half-space in spatial dimension  $d = 3$ .** Let

$$\Omega := (0, +\infty) \times \mathbf{R}^2 ; \quad (4.38)$$

this is the subcase of the general setting (4.1) corresponding to  $d = 3$  and

$$d_1 = 1 , \quad d_2 = 2 . \quad (4.39)$$

With the above choices, the symmetric group appearing in the general framework of subsection 4.5 consists of the sole identity ( $S_{d_1} = S_1 = \{id\}$ ). We have  $\mathbf{x}_1 = (x_1^1) \equiv x^1$  (and the analogous relations for  $\mathbf{y}_1$ ); besides,  $\mathfrak{N}_{0,id}(\mathbf{x}_1, \mathbf{y}_1) = |x^1 + y^1|$  and  $\mathfrak{N}_{1,id}(\mathbf{x}_1, \mathbf{y}_1) = |x^1 - y^1|$ . Using the relations (4.17-4.19), (2.16-2.18) and Eq.s (4.15) (4.16) and (4.37), with some simple algebraic manipulations we obtain the following expressions for the non-vanishing components of the regularized stress-energy VEV (where  $\mathbf{x} = (x^1, x^2, x^3)$ ):

$$\begin{aligned} \langle 0 | \widehat{T}_{00}^u(\mathbf{x}) | 0 \rangle = & - \frac{m^4}{32\pi^{3/2} \Gamma(\frac{u+1}{2})} \left( \frac{m}{\kappa} \right)^{-u} \left[ (1-u) \Gamma\left(\frac{u-4}{2}\right) + \right. \\ & \left. - 2^{5-\frac{u}{2}} \alpha_1 \left( \left( \frac{1}{4} - \xi \right) \mathfrak{G}_{\frac{u-2}{2}}(z^2) + \left( \frac{1}{2} - (3-u)\xi \right) \mathfrak{G}_{\frac{u-4}{2}}(z^2) \right) \right]_{z=2mx^1} ; \end{aligned} \quad (4.40)$$

$$\langle 0 | \widehat{T}_{11}^u(\mathbf{x}) | 0 \rangle = \frac{m^4}{32\pi^{3/2} \Gamma(\frac{u+1}{2})} \left( \frac{m}{\kappa} \right)^{-u} \Gamma\left(\frac{u-4}{2}\right) ; \quad (4.41)$$

$$\begin{aligned} \langle 0 | \widehat{T}_{22}^u(\mathbf{x}) | 0 \rangle = \langle 0 | \widehat{T}_{33}^u(\mathbf{x}) | 0 \rangle = & \frac{m^4}{32\pi^{3/2} \Gamma(\frac{u+1}{2})} \left( \frac{m}{\kappa} \right)^{-u} \left[ \Gamma\left(\frac{u-4}{2}\right) + \right. \\ & \left. - 2^{5-\frac{u}{2}} \alpha_1 \left( \left( \frac{1}{4} - \xi \right) \mathfrak{G}_{\frac{u-2}{2}}(z^2) + \left( \frac{2-u}{4} - (3-u)\xi \right) \mathfrak{G}_{\frac{u-4}{2}}(z^2) \right) \right]_{z=2mx^1} . \end{aligned} \quad (4.42)$$

The above expressions are easily seen to give the meromorphic continuation of the regularized stress-energy VEV to the whole complex plane, with poles determined by terms with the gamma function. In particular, all the above components have a simple pole in  $u = 0$ ; thus, we follow the extended version of the zeta approach and define the renormalized quantities to be the regular parts in  $u = 0$ . Recalling again that Eq. (2.14) gives

$$\xi_3 = \frac{1}{6} ,$$

we write the final results in the form (2.13), obtaining

$$\begin{aligned} \langle 0 | \widehat{T}_{00}(\mathbf{x}) | 0 \rangle_{ren} = & \frac{m^4}{384\pi^2} \left( 3 \left( 4 \ln \left( \frac{m}{2\kappa} \right) + 1 \right) + 32 \alpha_1 \frac{K_1(2mx^1)}{2mx^1} \right) + \\ & - \alpha_1 \left( \xi - \frac{1}{6} \right) \frac{m^4}{\pi^2} \left( \frac{(2mx^1)K_1(2mx^1) + 3K_2(2mx^1)}{(2mx^1)^2} \right) ; \end{aligned} \quad (4.43)$$

$$\langle 0 | \widehat{T}_{11}(\mathbf{x}) | 0 \rangle_{ren} = -\frac{m^4}{128\pi^2} \left( 4 \ln \left( \frac{m}{2\kappa} \right) - 3 \right) ; \quad (4.44)$$

$$\begin{aligned} \langle 0 | \widehat{T}_{22}(\mathbf{x}) | 0 \rangle_{ren} = \langle 0 | \widehat{T}_{33}(\mathbf{x}) | 0 \rangle_{ren} = & -\frac{m^4}{384\pi^2} \left( 3 \left( 4 \ln \left( \frac{m}{2\kappa} \right) - 3 \right) + 32 \alpha_1 \frac{K_1(2mx^1)}{2mx^1} \right) + \\ & - \alpha_1 \left( \xi - \frac{1}{6} \right) \frac{m^4}{\pi^2} \left( \frac{(2mx^1)K_1(2mx^1) + 3K_2(2mx^1)}{(2mx^1)^2} \right) . \end{aligned} \quad (4.45)$$

Let us comment briefly on the above results. Firstly, note that the renormalized VEV of the stress-energy tensor does not depend explicitly on the spatial coordinates  $x^2, x^3$ ; this comes as no surprise due to the homogeneity with respect to these variables of the spatial configuration considered. Besides, in agreement with the general results of subsection 4.6, the components  $\langle 0 | \widehat{T}_{11}^u | 0 \rangle$ ,  $\langle 0 | \widehat{T}_{11} | 0 \rangle_{ren}$  are constant and the two alternative definitions for the pressure on the plane  $\pi_1 = \{x_1 = 0\}$  (see Eq.s (4.27) (4.28)) give the same result; more explicitly, we obtain

$$p_i^{ren} = -\delta_{i1} \langle 0 | \widehat{T}_{11} | 0 \rangle_{ren} \quad (i \in \{1, 2, 3\}) . \quad (4.46)$$

In conclusion, let us make a comparison with the results derived in [2] for the configuration with a single plane in arbitrary spatial dimension (to be considered here with  $d = 3$ ); therein the attention is restricted to the “minimal” ( $\xi = 0$ ) and “conformal” ( $\xi = 1/6$ ) settings. In both cases the results derived here are found to agree with those reported in [2] (let us remark that the mass scale  $\kappa$  employed here does not coincide with the one considered therein; the latter is proportional, via a numerical coefficient, to  $m^2/\kappa$ ).

**4.9 The rectangular wedge.** Let us pass to the case of a wedge in  $\mathbf{R}^3$ , bounded by two perpendicular half-planes; this is represented as

$$\Omega := (0, +\infty)^2 \times \mathbf{R} , \quad (4.47)$$

corresponding to the general framework (4.1) with  $d = 3$  and

$$d_1 = 2 , \quad d_2 = 1 . \quad (4.48)$$

In passing, let us mention that the rectangular wedge model is also considered by Actor and Bender [2], for arbitrary spatial dimension; yet, these author restrict the attention to a massless field, a case we discuss in the next subsection.

In the present setting, the symmetric group ( $S_{d_1} = S_2$ ) of subsection 4.5 consists of two elements, i.e., the identity  $id$  and the exchange  $\mathbf{p}$ :

$$S_{d_1} \equiv S_2 = \{id, \mathbf{p}\} , \quad id(1) = 1 , \quad id(2) = 2 , \quad \mathbf{p}(1) = 2 , \quad \mathbf{p}(2) = 1 . \quad (4.49)$$

Moreover, with  $\mathbf{x}_1 = (x_1^1, x_1^2) \equiv (x^1, x^2)$  and the analogous relations for  $\mathbf{y}_1$ , we have

$$\begin{aligned} \mathfrak{N}_{0,\sigma}(\mathbf{x}_1, \mathbf{y}_1) &= \left( (x^1+y^1)^2 + (x^2+y^2)^2 \right)^{1/2} \\ \mathfrak{N}_{2,\sigma}(\mathbf{x}_1, \mathbf{y}_1) &= \left( (x^1-y^1)^2 + (x^2-y^2)^2 \right)^{1/2} \end{aligned} \quad \text{for } \sigma \in S_2 = \{id, \mathbf{p}\} ; \quad (4.50)$$

$$\mathfrak{N}_{1,id}(\mathbf{x}_1, \mathbf{y}_1) = \left( (x^1-y^1)^2 + (x^2+y^2)^2 \right)^{1/2} , \quad \mathfrak{N}_{1,\mathbf{p}}(\mathbf{x}_1, \mathbf{y}_1) = \left( (x^1+y^1)^2 + (x^2-y^2)^2 \right)^{1/2} .$$

Also in this case, we can use the relations (2.16-2.18), (4.17-4.19) and the identities (4.15) (4.16) and (4.37) to deduce expressions for the non-vanishing components of the regularized stress-energy VEV. More precisely, we obtain the following (with  $\mathbf{x} = (x^1, x^2, x^3)$ )

$$\begin{aligned} \langle 0 | \widehat{T}_{00}^u(\mathbf{x}) | 0 \rangle &= - \frac{m^4}{32\pi^{3/2} \Gamma(\frac{u+1}{2})} \left( \frac{m}{\kappa} \right)^{-u} \left[ (1-u) \Gamma\left(\frac{u-4}{2}\right) + \right. \\ &\quad - 2^{5-\frac{u}{2}} \sum_{i=1,2} \alpha_i \left( \left( \frac{1}{2} - (3-u)\xi \right) \mathfrak{G}_{\frac{u-4}{2}}(z^2) + \left( \frac{1}{4} - \xi \right) \mathfrak{G}_{\frac{u-2}{2}}(z^2) \right)_{z=2mx^i} + \\ &\quad \left. - 2^{5-\frac{u}{2}} \alpha_1 \alpha_2 \left( \left( \frac{1}{4} - (2-u)\xi \right) \mathfrak{G}_{\frac{u-4}{2}}(z^2) + \left( \frac{1}{4} - \xi \right) \mathfrak{G}_{\frac{u-2}{2}}(z^2) \right)_{z=2m\sqrt{(x^1)^2+(x^2)^2}} \right] ; \end{aligned} \quad (4.51)$$

$$\langle 0|\widehat{T}_{ij}^u(\mathbf{x})|0\rangle = \langle 0|\widehat{T}_{ji}^u(\mathbf{x})|0\rangle = \frac{m^4}{32\pi^{3/2}\Gamma(\frac{u+1}{2})} \left(\frac{m}{\kappa}\right)^{-u} \left[ \delta_{ij} \Gamma\left(\frac{u-4}{2}\right) + \right. \quad (4.52)$$

$$\begin{aligned} & - 2^{5-\frac{u}{2}} \alpha_{\mathfrak{p}(j)} \delta_{ij} \left( \left( \frac{2-u}{4} - (3-u)\xi \right) \mathfrak{G}_{\frac{u-4}{2}}(z^2) + \left( \frac{1}{4} - \xi \right) \mathfrak{G}_{\frac{u-2}{2}}(z^2) \right)_{z=2mx^{\mathfrak{p}(j)}} + \\ & - 2^{5-\frac{u}{2}} \alpha_1 \alpha_2 \left( \frac{1}{4} - \xi \right) \left( \delta_{ij} \left( (3-u) \mathfrak{G}_{\frac{u-4}{2}}(z^2) + \mathfrak{G}_{\frac{u-2}{2}}(z^2) \right) + \right. \\ & \quad \left. - 4 m^2 x^i x^j \mathfrak{G}_{\frac{u-6}{2}}(z^2) \right)_{z=2m\sqrt{(x^1)^2+(x^2)^2}} \Big] \quad \text{for } i, j \in \{1, 2\}; \end{aligned}$$

$$\langle 0|\widehat{T}_{33}^u(\mathbf{x})|0\rangle = \frac{m^4}{32\pi^{3/2}\Gamma(\frac{u+1}{2})} \left(\frac{m}{\kappa}\right)^{-u} \left[ \Gamma\left(\frac{u-4}{2}\right) + \right. \quad (4.53)$$

$$\begin{aligned} & - 2^{5-\frac{u}{2}} \sum_{i=1,2} \alpha_i \left( \left( \frac{2-u}{4} - (3-u)\xi \right) \mathfrak{G}_{\frac{u-4}{2}}(z^2) + \left( \frac{1}{4} - \xi \right) \mathfrak{G}_{\frac{u-2}{2}}(z^2) \right)_{z=2mx^i} + \\ & - 2^{5-\frac{u}{2}} \alpha_1 \alpha_2 \left( \left( \frac{1-u}{4} - (2-u)\xi \right) \mathfrak{G}_{\frac{u-4}{2}}(z^2) + \left( \frac{1}{4} - \xi \right) \mathfrak{G}_{\frac{u-2}{2}}(z^2) \right)_{z=2m\sqrt{(x^1)^2+(x^2)^2}} \Big]. \end{aligned}$$

The above expressions give the meromorphic continuation of the regularized stress-energy VEV to the whole complex plane, with poles determined by terms with the gamma function. In particular all components (4.51-4.53) have a simple pole in  $u = 0$ , and we must resort to the extended zeta approach taking again the regular parts at  $u = 0$ . Recalling once more that Eq. (2.14) gives

$$\xi_3 = \frac{1}{6},$$

hereafter we report separately the conformal and non-conformal parts of each component of the renormalized stress-energy VEV; these are

$$\begin{aligned} \langle 0|\widehat{T}_{00}^\diamond(\mathbf{x})|0\rangle_{ren} = \frac{m^4}{384\pi^2} \left[ 3 \left( 4 \ln\left(\frac{m}{2\kappa}\right) + 1 \right) + 32 \sum_{i=1,2} \alpha_i \left( \frac{K_1(z)}{z} \right)_{z=2mx^i} + \right. \\ \left. + 32 \alpha_1 \alpha_2 \left( \frac{z K_1(z) - K_2(z)}{z^2} \right)_{z=2m\sqrt{(x^1)^2+(x^2)^2}} \right], \end{aligned} \quad (4.54)$$

$$\begin{aligned} \langle 0|\widehat{T}_{00}^\blacksquare(\mathbf{x})|0\rangle_{ren} = - \frac{m^4}{\pi^2} \left[ \sum_{i=1,2} \alpha_i \left( \frac{z K_1(z) + 3K_2(z)}{z^2} \right)_{z=2mx^i} + \right. \\ \left. + \alpha_1 \alpha_2 \left( \frac{z K_1(z) + 2K_2(z)}{z^2} \right)_{z=2m\sqrt{(x^1)^2+(x^2)^2}} \right]; \end{aligned} \quad (4.55)$$

$$\begin{aligned}
\langle 0|\widehat{T}_{ij}^\diamond(\mathbf{x})|0\rangle_{ren} = & -\frac{m^4}{384\pi^2} \left[ 3\delta_{ij} \left( 4\ln\left(\frac{m}{2\kappa}\right) - 3 \right) + 32\alpha_{\mathfrak{p}(j)}\delta_{ij} \left( \frac{K_1(z)}{z} \right)_{z=2mx^{\mathfrak{p}(j)}} + \right. \\
& \left. + 32\alpha_1\alpha_2 \left( \delta_{ij} \frac{zK_1(z)+3K_2(z)}{z^2} - 4m^2x^ix^j \frac{K_3(z)}{z^3} \right)_{z=2m\sqrt{(x^1)^2+(x^2)^2}} \right] \\
& \text{for } i, j \in \{1, 2\} ; \tag{4.56}
\end{aligned}$$

$$\begin{aligned}
\langle 0|\widehat{T}_{ij}^\blacksquare(\mathbf{x})|0\rangle_{ren} = & \frac{m^4}{\pi^2} \left[ \alpha_{\mathfrak{p}(j)}\delta_{ij} \left( \frac{zK_1(z)+3K_2(z)}{z^2} \right)_{z=2mx^{\mathfrak{p}(j)}} + \right. \\
& \left. + \alpha_1\alpha_2 \left( \delta_{ij} \frac{zK_1(z)+3K_2(z)}{z^2} - 4m^2x^ix^j \frac{K_3(z)}{z^3} \right)_{z=2m\sqrt{(x^1)^2+(x^2)^2}} \right] \\
& \text{for } i, j \in \{1, 2\} ; \tag{4.57}
\end{aligned}$$

$$\begin{aligned}
\langle 0|\widehat{T}_{33}^\diamond(\mathbf{x})|0\rangle_{ren} = & -\frac{m^4}{384\pi^2} \left[ 3 \left( 4\ln\left(\frac{m}{2\kappa}\right) - 3 \right) + 32 \sum_{i=1,2} \alpha_i \left( \frac{K_1(z)}{z} \right)_{z=2mx^i} + \right. \\
& \left. + 32\alpha_1\alpha_2 \left( \frac{zK_1(z)-K_2(z)}{z^2} \right)_{z=2m\sqrt{(x^1)^2+(x^2)^2}} \right] ; \tag{4.58}
\end{aligned}$$

$$\begin{aligned}
\langle 0|\widehat{T}_{33}^\blacksquare(\mathbf{x})|0\rangle_{ren} = & \frac{m^4}{\pi^2} \left[ \sum_{i=1,2} \alpha_i \left( \frac{zK_1(z)+3K_2(z)}{z^2} \right)_{z=2mx^i} + \right. \\
& \left. + \alpha_1\alpha_2 \left( \frac{zK_1(z)+2K_2(z)}{z^2} \right)_{z=2m\sqrt{(x^1)^2+(x^2)^2}} \right] . \tag{4.59}
\end{aligned}$$

As was to be expected, since the configuration (4.47) is invariant under translation along the  $x^3$  direction, none of the expressions (4.54-4.59) depends on the spatial coordinate  $x^3$ .

Let us now discuss the pressure at points in the half-plane  $\pi_1 = \{x^1 = 0, x^2 > 0\}$ ; note that, we are excluding points on the axis  $\zeta := \{x^1 = x^2 = 0\}$ . The two alternative definitions (4.27) (4.28) are easily seen to give the same result for the renormalized version of this quantity, in agreement with the general results of subsection 4.6: indeed, we can equivalently put  $x^1 = 0$  in Eq. (4.52) for  $\langle 0|\widehat{T}_{i1}^u(\mathbf{x})|0\rangle$  ( $i \in \{1, 2, 3\}$ ) and then analytically continue up to  $u = 0$ , or directly evaluate the renormalized expressions (4.56) (4.57) for  $\langle 0|\widehat{T}_{i1}(\mathbf{x})|0\rangle_{ren}$  in  $x^1 = 0$ . In both ways, we obtain

$$\begin{aligned}
p_i^{ren}(\mathbf{x})\Big|_{\pi_1} = & \delta_{i1} \left[ \frac{m^4}{384\pi^2} \left( 3 \left( 4\ln\left(\frac{m}{2\kappa}\right) - 3 \right) + 32\alpha_2 \frac{(1+\alpha_1)zK_1(z)+3\alpha_1K_2(z)}{z^2} \right) + \right. \\
& \left. - \left( \xi - \frac{1}{6} \right) \frac{m^4}{\pi^2} (1+\alpha_1)\alpha_2 \frac{zK_1(z)+3K_2(z)}{z^2} \right]_{z=2mx^2} \quad (i \in \{1, 2, 3\}) . \tag{4.60}
\end{aligned}$$

Let us stress that, differently from all the configurations considered so far within this paper, in this case the renormalized pressure on the boundary depends in general on the parameter  $\xi$ ; more precisely, this happens whenever Neumann boundary conditions are imposed on the half-plane  $\pi_1$  (so that  $\alpha_1 = +1$ ).

Now, let us consider the axis  $\zeta = \{x^1 = x^2 = 0\}$ ; at any point of this axis the outer normal is ill defined, so that there is a basic obstruction to speaking of the pressure. However, we can discuss what happens if a point  $\mathbf{x} = (0, x^2, x^3) \in \pi_1$  moves towards the axis  $\zeta$ , i.e., if we consider the limit  $x^2 \rightarrow 0^+$ . In this limit  $p_i^{ren}(\mathbf{x})$  is found to diverge; more precisely, Eq. (4.60) and the known asymptotic behaviour of the Bessel function  $K_\nu$  near zero (see [27], p.252, Eq.10.30.2) give the following:

$$p_1^{ren}(\mathbf{x}) = O\left(\frac{1}{(x^2)^4}\right) \quad \text{for } \mathbf{x} \in \pi_1, x^2 \rightarrow 0^+ . \quad (4.61)$$

**4.10 The previous examples in the zero mass limit.** Let us first consider the half-space configuration (4.38) bounded by the plane  $\pi_1$ , analysed in subsection 4.8. The expressions (4.43-4.45) for the components of  $\langle 0|\hat{T}_{\mu\nu}|0\rangle_{ren}$  give, in the limit  $m \rightarrow 0^+$ ,

$$\langle 0|\hat{T}_{\mu\nu}(\mathbf{x})|0\rangle_{ren}\Big|_{\mu,\nu=0,1,2,3} = \left(\xi - \frac{1}{6}\right) \frac{3\alpha_1}{8\pi^2(x^1)^4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad (4.62)$$

as for the pressure on a point  $\mathbf{x} \in \pi_1$ , starting with Eq. (4.46) and taking the limit  $m \rightarrow 0^+$ , it is trivial to infer

$$p_i^{ren}(\mathbf{x}) = 0 \quad (i \in \{1, 2, 3\}) . \quad (4.63)$$

Let us pass to the case of the rectangular wedge (see subsection 4.9). Eq.s (4.54-4.59) for the renormalized stress-energy VEV give, in the limit  $m \rightarrow 0^+$ ,

$$\begin{aligned} \langle 0|\hat{T}_{\mu\nu}(\mathbf{x})|0\rangle_{ren}\Big|_{\mu,\nu=0,1,2,3} &= \frac{\alpha_1\alpha_2}{96\pi^2\rho^4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & A_1(\mathbf{x}) & B(\mathbf{x}) & 0 \\ 0 & B(\mathbf{x}) & A_2(\mathbf{x}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \\ &- \left(\xi - \frac{1}{6}\right) \left[ \frac{\alpha_1\alpha_2}{8\pi^2\rho^4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & A_1(\mathbf{x}) & B(\mathbf{x}) & 0 \\ 0 & B(\mathbf{x}) & A_2(\mathbf{x}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{3}{8\pi^2} \begin{pmatrix} -C_0(\mathbf{x}) & 0 & 0 & 0 \\ 0 & C_1(\mathbf{x}) & 0 & 0 \\ 0 & 0 & C_2(\mathbf{x}) & 0 \\ 0 & 0 & 0 & C_0(\mathbf{x}) \end{pmatrix} \right] ; \\ A_i(\mathbf{x}) &:= 1 - \frac{4(x^{\mathfrak{p}(i)})^2}{\rho^2} , \quad C_i(\mathbf{x}) := \frac{\alpha_{\mathfrak{p}(i)}}{(x^{\mathfrak{p}(i)})^4} , \quad \text{for } i = 1, 2 ; \end{aligned} \quad (4.64)$$

$$B(\mathbf{x}) := \frac{4x^1x^2}{\rho^2} , \quad C_0(\mathbf{x}) := \frac{\alpha_1}{(x^1)^4} + \frac{\alpha_2}{(x^2)^4} + \frac{\alpha_1\alpha_2}{\rho^4} , \quad \rho := \sqrt{(x^1)^2 + (x^2)^2}$$



(recall that  $\mathbf{p}(1) = 2$ ,  $\mathbf{p}(2) = 1$ ). Next, consider the expression (4.60) for the pressure acting on a point  $\mathbf{x}$  in the half-plane  $\pi_1 = \{x^1 = 0, x^2 > 0\}$ , for  $m > 0$ ; using the asymptotic behaviour of the Bessel function  $K_\nu$  near zero (see [27], p.252, Eq.10.30.2) we infer, in the limit  $m \rightarrow 0^+$ ,

$$p_i^{ren}(\mathbf{x}) = \delta_{i1} \left[ \frac{\alpha_1 \alpha_2}{32\pi^2(x^2)^4} - \left( \xi - \frac{1}{6} \right) \frac{3(1+\alpha_1)\alpha_2}{8\pi^2(x^2)^4} \right] \quad (i \in \{1, 2, 3\}) . \quad (4.65)$$

Notice that, as for the massive analogue (4.60), the above expression for the renormalized pressure depends explicitly on  $\xi$  if we assume Neumann boundary conditions on  $\pi_1$  (so that  $\alpha_1 = +1$ ).

In passing, let us remark that both results (4.63) and (4.65) for the pressure on  $\pi_1$  could be determined equivalently via the prescription (2.58); according to the latter, we should have first considered the renormalized stress-energy VEV inside the corresponding spatial domain in the limit of zero mass (see Eq.s (4.62) (4.64)) and then move to the boundary (half-)plane  $\pi_1$ , i.e., take the limit  $x^1 \rightarrow 0^+$  <sup>(12)</sup>.

Let us comment briefly on the construction described above, namely, that of taking the zero mass limit ( $m \rightarrow 0^+$ ) of the renormalized results (4.43-4.45) (4.46) and (4.54-4.59) (4.60) for the theory with a massive field. As a matter of fact, this procedure corresponds to studying the case of a massless scalar field (in the same spatial configurations) with the technique of subsection 2.14. Indeed, in the massless case ( $m = 0$ ) the spectrum of the fundamental operator  $\mathcal{A} = -\Delta$  is  $[0, +\infty)$ , for both the settings (4.38) and (4.47); according to the framework of the cited subsection, we could treat these cases using the deformed operator  $\mathcal{A}_\varepsilon := \mathcal{A} + \varepsilon^2$  (see Eq. (5.14)), and eventually taking the limit  $\varepsilon \rightarrow 0^+$ . On the other hand, if  $\varepsilon$  is identified with  $m$ , we recover the present constructions <sup>(13)</sup>.

Summing up: Eq.s (4.62-4.63) and (4.64-4.65) yield the renormalized VEVs of the stress-energy tensor and pressure for a  $d = 3$  massless field, respectively confined within a half-space and a rectangular wedge, fulfilling either Dirichlet ( $\alpha_i = -1$ ) or Neumann ( $\alpha_i = +1$ ) boundary conditions.

We point out that, in the Dirichlet case, the above results are found to agree with those derived by Actor and Bender [2] for  $\xi = 0$  and  $\xi = 1/6$ , via a different version of the zeta approach (also involving, essentially, a subtraction of divergent contributions; see subsections 3.1.1 and 4.1 of the cited paper, setting  $d = 3$  therein).

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<sup>12</sup>Notice that not all of the components in Eq.s (4.62) (4.64) are finite for  $x^1 \rightarrow 0^+$ , but only those involved in the computation of the pressure on  $\pi_1$ ; for example, in both cases we have

$$\lim_{x^1 \rightarrow 0^+} \langle 0 | \hat{T}_{22}(\mathbf{x}) | 0 \rangle_{ren} = \infty .$$

<sup>13</sup>The same comments could be made in general for any spatial dimension  $d$  and for any number  $d_1$  of faces.

Let us also mention that the massless half-space configuration in the case of Dirichlet boundary conditions was considered as well in our previous work [17]; therein the same results were obtained starting from the renormalized stress-energy VEV of a massless field between parallel planes, and taking the limit of infinite distance between the planes (see Eq. (5.7) in [17]; after an exchange  $1 \leftrightarrow 3$  in the coordinate labels this becomes the present Eq. (4.62), with  $\alpha_1 = -1$ ).

In the next section we show that the renormalized expressions (4.62-4.65), here deduced as the zero mass limit of a massive theory, can be obtained equivalently as particular cases of the theory of a massless scalar field confined within two half-planes forming an angle  $\alpha$  of arbitrary width; more precisely, the present configurations with one single plane and two orthogonal planes correspond, respectively, to the limits  $\alpha \rightarrow \pi$  and  $\alpha \rightarrow \pi/2$ .

## 5 The case of massless field in a 3-dimensional wedge

**5.1 Introducing the problem for arbitrary boundary conditions.** In this section we consider the case of a scalar field (with no external forces) confined within a 3-dimensional wedge, meaning that the spatial domain  $\Omega$  is the portion of the space  $\mathbf{R}^3$  enclosed by two half-planes  $\pi_0, \pi_\alpha$  forming an angle  $\alpha \in (0, 2\pi]$ ; we assume  $\pi_0 = \{x^2=0, x^1 \geq 0\}$ . Suitable boundary conditions will be specified in the following.

We choose to confine our attention to the massless case ( $V = 0$ ) since, in this case, we are able to perform the explicit computations with a moderate effort for arbitrary values of the angle  $\alpha \in (0, 2\pi]$ ; in the massive case ( $V = m^2$ ) we could give an exhaustive analysis for rational values of  $\alpha/\pi$  but this would require a big computational effort and produce cumbersome expressions for the final results.

We already pointed out in the Introduction that the present framework has been previously considered by Dowker [12, 13], Deutsch and Candelas [11], Saharian et al. [29, 31] and by Fulling et al. [23], amongst other; a more detailed comparison between our results and these works will be performed in subsections 5.6-5.9.

In passing let us notice that, in case of either Dirichlet or Neumann boundary conditions, the spatial domain under analysis corresponds for  $\alpha = \pi$  and  $\alpha = \pi/2$ , respectively, to the configurations with a boundary made of a single plane and of two orthogonal half-planes. Besides, let us also stress that for  $\alpha = 2\pi$  the two half-planes  $\pi_0, \pi_\alpha$  overlap; because of this, for Dirichlet or Neumann boundary conditions, the boundary consists of a single half-plane, while, in the case of periodic boundary conditions, the spatial domain  $\Omega$  can be identified with  $\mathbf{R}^3$ . So, in the latter case one is actually considering a massless scalar field on the whole Minkowski spacetime.

We will comment further on each specific case in the next subsections. In particular, we will show that in the cases with  $\alpha = \pi$  and  $\alpha = \pi/2$  one recovers the results of subsection 4.10.

In order to deal with the present configuration, it is advisable to pass to the system of cylindrical coordinates

$$\mathbf{x} \mapsto \mathbf{q}(\mathbf{x}) \equiv (\rho(\mathbf{x}), \theta(\mathbf{x}), z(\mathbf{x})) \in (0, +\infty) \times (0, 2\pi) \times \mathbf{R} , \quad (5.1)$$

whose inverse will be written  $\mathbf{q} \mapsto \mathbf{x}(\mathbf{q})$ . These coordinates are chosen so that  $z(\mathbf{x}) = x^3$  and the boundary  $\partial\Omega$  corresponds to the limit values  $\theta = 0$  and  $\theta = \alpha$ ; the spatial line element reads

$$d\ell^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2 . \quad (5.2)$$

In order to avoid clumsy notations, given any function  $\Omega \rightarrow Y$ ,  $\mathbf{x} \mapsto f(\mathbf{x})$  (with  $Y$  any set), the composition  $\mathbf{q} \in \mathbf{q}(\Omega) \mapsto f(\mathbf{x}(\mathbf{q}))$  will be written as  $\mathbf{q} \mapsto f(\mathbf{q})$ .

Since curvilinear coordinates are being employed, one should refer to the framework of subsection 2.15. Writing  $\rho, \theta, z$  for the coordinate labels, we see that the only non-vanishing Christoffel symbols associated to the line element (5.2) are  $\gamma_{\theta\theta}^\rho = -\rho$  and  $\gamma_{\rho\theta}^\theta = \gamma_{\theta\rho}^\theta = \frac{1}{\rho}$ ; so, the second order covariant derivatives of any scalar function  $f$  are given by

$$\begin{aligned} \nabla_{\rho\rho}f &= \partial_{\rho\rho}f , & \nabla_{\rho\theta}f &= \partial_{\rho\theta}f - \frac{1}{\rho} \partial_\theta f , & \nabla_{\rho z}f &= \partial_{\rho z}f , \\ \nabla_{\theta\theta}f &= \partial_{\theta\theta}f + \rho \partial_\rho f , & \nabla_{\theta z}f &= \partial_{\theta z}f , & \nabla_{zz}f &= \partial_{zz}f . \end{aligned} \quad (5.3)$$

In conclusion, let us stress that the configuration under analysis could be dealt with as a slab configuration where  $\Omega = \Omega_1 \times \mathbf{R}$ , and  $\Omega_1 \subset \mathbf{R}^2$  corresponds to  $(0, +\infty) \times (0, \alpha)$  in terms of the coordinates  $(\rho, \theta)$ ; yet, this approach is not convenient. In fact, if one works directly on the 3-dimensional spatial domain  $\Omega$  (for any of the boundary conditions to be considered in the following), the modified cylinder kernel  $\tilde{T}$  defined in Eq. (2.28) associated to the fundamental operator  $\mathcal{A} = -\Delta$  can be expressed in terms of elementary functions (moreover, it is meromorphic in the variable  $\mathbf{t}$  on the whole complex plane). On the contrary, to treat the problem as a slab configuration we should use the analogous kernel  $\tilde{T}$  for the reduced operator  $\mathcal{A}_1$  on  $\Omega_1$ , or any other integral kernel related to the latter; these kernels do not possess simple expressions (only integral representations are available), so that the whole analysis would become a lot more involved.

**5.2 The Dirichlet kernel.** For any of the boundary conditions to be considered in the following, it turns out that the spectrum of  $\mathcal{A}$  is  $[0, +\infty)$ . Since  $\{0\}$  is a non-isolated point of the spectrum, we must resort to the methods discussed in subsection 2.14 to determine the renormalized Dirichlet kernel (and its derivatives); we

regularize the theory using the deformed operator  $\mathcal{A}_\varepsilon = (\sqrt{\mathcal{A}} + \varepsilon)^2$  (see Eq. (5.15)), whose choice is found, *a posteriori*, to be more effective from the computational viewpoint. In the sequel (see subsections 5.6-5.9) we derive the explicit expression of the modified cylinder kernel  $\tilde{T}$  of  $\mathcal{A}$ , for several types of boundary conditions; in any case  $\tilde{T}$  is found to be a meromorphic function of  $\mathfrak{t}$ , decreasing faster than  $\mathfrak{t}^{-1}$  for  $\Re \mathfrak{t} \rightarrow +\infty$ ; this allows us to proceed as explained in subsection 2.14. In this way we obtain, for the Dirichlet kernels and its derivatives, these renormalized expressions at  $s = \pm 1/2$ , respectively (see Eq. (2.73)):

$$D_{-\frac{1}{2}}(\mathbf{q}, \mathbf{p}) \Big|_{\mathbf{p}=\mathbf{q}} = \text{Res} \left( 2 \mathfrak{t}^{-3} \tilde{T}(\mathfrak{t}; \mathbf{q}, \mathbf{p}) \Big|_{\mathbf{p}=\mathbf{q}}; 0 \right) ; \quad (5.4)$$

$$\nabla_{zw} D_{+\frac{1}{2}}(\mathbf{q}, \mathbf{p}) \Big|_{\mathbf{p}=\mathbf{q}} = \text{Res} \left( \mathfrak{t}^{-1} \nabla_{vw} \tilde{T}(\mathfrak{t}; \mathbf{q}, \mathbf{p}) \Big|_{\mathbf{p}=\mathbf{q}}; 0 \right) , \quad (5.5)$$

for  $v, w$  any two cylindrical coordinates .

Here and in the sequel, we are using the following notations:

$$\begin{aligned} \mathbf{q} &\equiv (\rho, \theta, z) , \quad \mathbf{p} \equiv (\rho', \theta', z') ; \\ f(\mathbf{x}(\mathbf{q}), \mathbf{x}(\mathbf{p})) &\equiv f(\mathbf{q}, \mathbf{p}) \quad \text{for any } f : \Omega \times \Omega \rightarrow Y \text{ (} Y \text{ a set)} . \end{aligned} \quad (5.6)$$

**5.3 The stress-energy tensor.** Relations (5.4) (5.5), along with Eq.s (2.16-2.18), can be used to obtain the following expressions for the renormalized VEV of the stress-energy tensor:

$$\begin{aligned} \langle 0 | \widehat{T}_{00}(\mathbf{q}) | 0 \rangle_{ren} &= \text{Res} \left( \mathfrak{t}^{-3} \left[ \left( \frac{1}{2} + 2\xi \right) \tilde{T}(\mathfrak{t}; \mathbf{q}, \mathbf{p}) + \right. \right. \\ &\quad \left. \left. + \left( \frac{1}{4} - \xi \right) \mathfrak{t}^2 \partial^{q^\ell} \partial_{p^\ell} \tilde{T}(\mathfrak{t}; \mathbf{q}, \mathbf{p}) \right] \Big|_{\mathbf{p}=\mathbf{q}}; 0 \right) ; \end{aligned} \quad (5.7)$$

$$\begin{aligned} \langle 0 | \widehat{T}_{ij}(\mathbf{q}) | 0 \rangle_{ren} &= \langle 0 | \widehat{T}_{ji}(\mathbf{q}) | 0 \rangle_{ren} = \text{Res} \left( \mathfrak{t}^{-3} \left[ \left( \frac{1}{2} - 2\xi \right) \delta_{ij} \tilde{T}(\mathfrak{t}; \mathbf{q}, \mathbf{p}) + \right. \right. \\ &\quad \left. \left. + \mathfrak{t}^2 \left( \left( \frac{1}{4} - \frac{\xi}{2} \right) \partial_{q^i p^j} - \frac{\xi}{2} \nabla_{q^i q^j} - \left( \frac{1}{4} - \xi \right) \delta_{ij} \partial^{q^\ell} \partial_{p^\ell} \right) \tilde{T}(\mathfrak{t}; \mathbf{q}, \mathbf{p}) \right] \Big|_{\mathbf{p}=\mathbf{q}}; 0 \right) \end{aligned} \quad (5.8)$$

for  $i, j \in \{\rho, \theta, z\}$  .

In the following subsections we will first compute  $\tilde{T}$  and then explicitly evaluate the residues appearing in Eq.s (5.7) (5.8) for the cases where either Dirichlet, Neumann or periodic boundary conditions are prescribed; we will present the final results in the form (2.13), recalling once again that Eq. (2.14) implies

$$\xi_3 = \frac{1}{6} . \quad (5.9)$$

**5.4 The boundary forces.** In order to discuss this topic, since the spectrum of  $\mathcal{A}$  contains  $\{0\}$  as a non-isolated point, we must resort once more to the framework of subsection 2.14; again, we choose to use the deformed operator  $\mathcal{A}_\varepsilon := (\sqrt{\mathcal{A}} + \varepsilon)^2$ . We can consider the two alternative definitions (2.66) (2.67) for the renormalized pressure on the boundary. Contrary to the configurations analized in the previous sections, in this case the two mentioned alternatives *do not yield the same result*. As an example, let us focus on the pressure acting on the half-plane  $\pi_\alpha$ . We indicate with  $\mathbf{n}(\mathbf{q})$  the unit outer normal at a point of this half-plane with coordinates  $\mathbf{q} = (\rho, \alpha, z)$ ; this has components  $(n^\rho, n^\theta, n^z)(\mathbf{q}) = (0, 1/\rho, 0)$ . On the one hand, prescription (2.66) corresponds to put, for  $i \in \{\rho, \theta, z\}$ ,

$$p_i^{ren}(\mathbf{q}) := \lim_{\varepsilon \rightarrow 0^+} \left( RP \Big|_{u=0} \langle 0 | \widehat{T}_{ij}^{\varepsilon u}(\mathbf{q}) | 0 \rangle n^j(\mathbf{q}) \right) = \frac{1}{\rho} \lim_{\varepsilon \rightarrow 0^+} RP \Big|_{u=0} \langle 0 | \widehat{T}_{i\theta}^{\varepsilon u}(\mathbf{q}) | 0 \rangle \quad (5.10)$$

(first consider the regularized stress-energy VEV on  $\pi_\alpha$ , then analytically continue at  $u = 0$  and finally take the limit  $\varepsilon \rightarrow 0^+$ ). Similarly to Eq. (5.8), the prescription (5.10) yields

$$p_i^{ren}(\mathbf{q}) = \frac{1}{\rho} \text{Res} \left( \mathfrak{t}^{-3} \left[ \left( \frac{1}{2} - 2\xi \right) \delta_{i\theta} \tilde{T}(\mathfrak{t}; \mathbf{q}, \mathbf{p}) + \right. \right. \\ \left. \left. + \mathfrak{t}^2 \left( \left( \frac{1}{4} - \frac{\xi}{2} \right) \partial_{q^i \theta'} - \frac{\xi}{2} \nabla_{q^i \theta} - \left( \frac{1}{4} - \xi \right) \delta_{i\theta} \partial^{q^\ell} \partial_{p^\ell} \right) \tilde{T}(\mathfrak{t}; \mathbf{q}, \mathbf{p}) \right] \Big|_{\mathbf{p}=\mathbf{q} \in \pi_\alpha}; 0 \right); \quad (5.11)$$

let us stress that in the above expression, we first perform the evaluation on the boundary and then compute the residue.

On the other hand, the alternative prescription (2.67) yields, for  $i \in \{\rho, \theta, z\}$ ,

$$p_i^{ren}(\mathbf{q}) := \left( \lim_{\mathbf{q}' \rightarrow \mathbf{q}, \mathbf{x}(\mathbf{q}') \in \Omega} \langle 0 | \widehat{T}_{ij}(\mathbf{q}') | 0 \rangle_{ren} \right) n^j(\mathbf{q}) = \\ = \frac{1}{\rho} \left( \lim_{\mathbf{q}' \rightarrow \mathbf{q}, \mathbf{x}(\mathbf{q}') \in \Omega} \langle 0 | \widehat{T}_{i\theta}(\mathbf{q}') | 0 \rangle_{ren} \right). \quad (5.12)$$

In the cases to be considered in subsections 5.6-5.8, it will be apparent that the explicit expressions obtained for the renormalized stress-energy VEV inside the wedge diverge when approaching the boundary, in such a way to make divergent the renormalized pressure defined by (5.12). Because of this, we will always refer to Eq. (5.11) to deal with the pressure on the boundary.

**5.5 Some remarks.** For the computation of  $\tilde{T}$  we will often refer to [23], where this kernel was already determined for the present configuration (but used in a different renormalization scheme, based on point splitting as an alternative to zeta regularization; we note that the kernel  $\overline{T}$  in [23] is the opposite of the kernel  $\tilde{T}$  considered here).

At the end of each subsection dealing with Dirichlet and/or Neumann boundary conditions, we will comment briefly on the results obtained for the renormalized stress-energy VEV (5.7-5.8) and pressure (5.12) in the cases  $\alpha = \pi$  and  $\alpha = \pi/2$ , respectively, describing a massless scalar field on a half-space and inside a wedge with orthogonal half-planes. Recall that these very same configurations were analysed as the zero mass limit of a corresponding massive theory in subsection 4.10; indeed, we will find that the same results obtained therein can be re-obtained from Eq.s (5.7-5.8) and (5.12), returning to the Cartesian coordinates  $x^1 = \rho \sin \theta$ ,  $x^2 = \rho \cos \theta$ ,  $x^3 = z$  and considering the appropriate transformation laws <sup>(14)</sup>.

**5.6 Dirichlet boundary conditions.** Let us first consider the case where the field fulfills Dirichlet boundary conditions on the half-planes  $\pi_0, \pi_\alpha$ . In this case a complete orthonormal system of (improper) eigenfunctions  $(F_k)_{k \in \mathcal{K}}$  of  $\mathcal{A} = -\Delta$ , with eigenvalues  $(\omega_k^2)_{k \in \mathcal{K}}$ , is given by

$$F_k(\mathbf{q}) := \sqrt{\frac{\omega}{\pi\alpha}} J_{\lambda_n}(\omega\rho) \sin(\lambda_n\theta) e^{ihz}, \quad \lambda_n := \frac{n\pi}{\alpha}, \quad (5.13)$$

$$\omega_k^2 := \omega^2 + h^2 \quad \text{for } k \equiv (n, \omega, h) \in \mathcal{K} \equiv \mathbf{N} \times (0, +\infty) \times \mathbf{R}$$

(here and elsewhere we are considering the set of positive integers with  $\mathbf{N} := \{1, 2, 3, \dots\}$ ;  $\mathcal{K}$  is endowed with the counting measure on  $\mathbf{N}$  times the standard Lebesgue measure on  $(0, +\infty) \times \mathbf{R}$ , meaning that  $\int_{\mathcal{K}} dk \equiv \sum_{n=1}^{+\infty} \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} dh$ ). The modified cylinder kernel  $\tilde{T}$  can be evaluated starting from its eigenfunction expansion (2.28), which in the present setting reads

$$\tilde{T}(\mathbf{t}; \mathbf{q}, \mathbf{p}) = \int_{\mathcal{K}} dk \frac{e^{-t\omega_k}}{\omega_k} F_k(\mathbf{q}) \overline{F_k}(\mathbf{p}) = \quad (5.14)$$

$$\frac{1}{\pi\alpha} \sum_{n=1}^{+\infty} \sin(\lambda_n\theta) \sin(\lambda_n\theta') \int_0^{+\infty} d\omega \omega J_{\lambda_n}(\omega\rho) J_{\lambda_n}(\omega\rho') \int_{-\infty}^{+\infty} dh \frac{e^{-t\sqrt{\omega^2+h^2}}}{\sqrt{\omega^2+h^2}} e^{ih(z-z')}.$$

With some effort, the integrals over  $h$  and  $\omega$  in the above expression can be explicitly evaluated to yield

$$\tilde{T}(\mathbf{t}; \mathbf{q}, \mathbf{p}) = \frac{1}{\pi\alpha \rho\rho' \sinh v} \sum_{n=1}^{+\infty} \sin(\lambda_n\theta) \sin(\lambda_n\theta') e^{-\lambda_n v}, \quad (5.15)$$

$$v := -\ln\left(\frac{r_+ - r_-}{r_+ + r_-}\right), \quad r_{\pm} := \sqrt{(\rho \pm \rho')^2 + (z - z')^2 + t^2}.$$

We refer to [23] for the detailed computations giving the above result; the notations  $v, r_{\pm}$  are mutuuated from this reference (also see [24]). The series in Eq. (5.15) can be

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<sup>14</sup>Clearly,  $\langle 0 | \widehat{T}_{\mu\nu}(\mathbf{q}) | 0 \rangle_{ren}$  and  $p_i^{ren}(\mathbf{q})$  transform, respectively, as a rank-two tensor and a vector.

re-expressed via four geometric series writing the trigonometric functions in terms of complex exponentials; in this way we obtain

$$\tilde{T}(\mathbf{t}; \mathbf{q}, \mathbf{p}) = \frac{1}{4\pi\alpha\rho\rho'\sinh v} \left( \frac{\cos(\frac{\pi}{\alpha}(\theta-\theta')) - e^{-\frac{\pi}{\alpha}v}}{\cosh(\frac{\pi}{\alpha}v) - \cos(\frac{\pi}{\alpha}(\theta-\theta'))} - \frac{\cos(\frac{\pi}{\alpha}(\theta+\theta')) - e^{-\frac{\pi}{\alpha}v}}{\cosh(\frac{\pi}{\alpha}v) - \cos(\frac{\pi}{\alpha}(\theta+\theta'))} \right). \quad (5.16)$$

Now, we resort to Eq.s (5.7-5.8) for the renormalized stress-energy VEV; evaluating the residues therein we obtain

$$\begin{aligned} & \langle 0 | \hat{T}_{\mu\nu}(\mathbf{q}) | 0 \rangle_{ren} \Big|_{\mu,\nu=0,\rho,\theta,z} = \\ & A(\mathbf{q}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3\rho^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \left( \xi - \frac{1}{6} \right) \begin{pmatrix} -(B(\mathbf{q})+C(\mathbf{q})) & 0 & 0 & 0 \\ 0 & B(\mathbf{q}) & -\rho E(\mathbf{q}) & 0 \\ 0 & -\rho E(\mathbf{q}) & \rho^2 C(\mathbf{q}) & 0 \\ 0 & 0 & 0 & B(\mathbf{q})+C(\mathbf{q}) \end{pmatrix}, \\ & A(\mathbf{q}) := \frac{\pi^4 - \alpha^4}{1440\pi^2\alpha^4\rho^4}, \quad B(\mathbf{q}) := \frac{9\pi^4 - 3\pi^2(2\pi^2 + \alpha^2)\sin^2(\frac{\pi\theta}{\alpha}) + \alpha^2(\pi^2 - \alpha^2)\sin^4(\frac{\pi\theta}{\alpha})}{24\pi^2\alpha^4\sin^4(\frac{\pi\theta}{\alpha})\rho^4}, \\ & C(\mathbf{q}) := \frac{3\pi^2 - (\pi^2 - \alpha^2)\sin^4(\frac{\pi\theta}{\alpha})}{8\pi^2\alpha^2\sin^2(\frac{\pi\theta}{\alpha})\rho^4}, \quad E(\mathbf{q}) := \frac{3\pi\cos(\frac{\pi\theta}{\alpha})}{8\alpha^3\sin^3(\frac{\pi\theta}{\alpha})\rho^4}. \end{aligned} \quad (5.17)$$

It can be easily checked that the above result agrees with the one derived by Saharian and Tarloian by means of point-splitting regularization in [31] (see, in particular, Section 3 therein); see also the former papers by Dowker et al. [12, 13] and by Deutsch and Candelas [11], where point-splitting regularization is used for the computation of the conformal stress-energy VEV alone.

Let us briefly comment on the explicit expression (5.17) obtained for the renormalized VEV  $\langle 0 | \hat{T}_{\mu\nu}(\mathbf{q}) | 0 \rangle_{ren}$ . First of all, notice that the function  $A(\mathbf{q})$  multiplying the conformal part of the renormalized VEV is positive for  $\alpha < \pi$ , negative for  $\alpha > \pi$  and vanishes for  $\alpha = \pi$ . Next, let us stress that both the conformal and non-conformal parts diverge quartically in  $\rho$  in the proximity of the axis  $\{\rho = 0\}$ . The non-conformal part also diverges near the half-planes  $\pi_0, \pi_\alpha$ , that is for  $\theta \rightarrow 0, \alpha$ ; because of this, the pressure on these half-planes evaluated according to Eq. (5.12) is infinite. On the other hand, the alternative definition (5.10-5.11) (first move to the boundary, and then take the analytic continuation) gives a finite pressure on  $\pi_\alpha$  with components

$$p_i^{ren}(\mathbf{q}) = -\delta_{i\theta} \frac{\pi^4 - \alpha^4}{480\pi^2\alpha^4\rho^3}. \quad (5.18)$$

To conclude, we consider the special cases  $\alpha = \pi$  and  $\alpha = \pi/2$  (a space domain bounded by a plane or by two perpendicular half-planes), and compare the present

results with the ones of subsection 4.10. This can be done with the procedure outlined in subsection 5.5 (i.e., returning to Cartesian coordinates via the appropriate transformation rules for tensor coefficients). Indeed, the expressions (5.17) (5.18) (giving the renormalized stress-energy VEV and pressure) are easily seen to yield for  $\alpha = \pi$  and  $\alpha = \pi/2$ , respectively, Eq.s (4.62) (4.63) with  $\alpha_1 = -1$ , and Eq.s (4.64) (4.65) with  $\alpha_1 = \alpha_2 = -1$ .

**5.7 Dirichlet-Neumann boundary conditions.** Let us now pass to the analysis of the wedge configuration where Dirichlet and Neumann boundary conditions are prescribed, respectively, on the half-planes  $\pi_0$  and  $\pi_\alpha$ ; let us point out that, to the best of our knowledge, this case has never been considered before in the literature. A complete orthonormal system of (improper) eigenfunctions  $(F_k)_{k \in \mathcal{K}}$  of the fundamental operator  $\mathcal{A}$ , with eigenvalues  $(\omega_k^2)_{k \in \mathcal{K}}$ , is given by

$$F_k(\mathbf{q}) := \sqrt{\frac{\omega}{\pi\alpha}} J_{\lambda_n}(\omega\rho) \sin(\lambda_n\theta) e^{ihz}, \quad \lambda_n := \left(n + \frac{1}{2}\right) \frac{\pi}{\alpha}, \quad (5.19)$$

$$\omega_k^2 := \omega^2 + h^2 \quad \text{for } k \equiv (n, \omega, h) \in \mathcal{K} \equiv \mathbf{N}_0 \times (0, +\infty) \times \mathbf{R}$$

( $\mathbf{N}_0 := \{0, 1, 2, \dots\}$  is the set of non-negative integers; the measure on the label space  $\mathcal{K}$  is such that  $\int_{\mathcal{K}} dk \equiv \sum_{n=0}^{+\infty} \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} dh$ ). Resorting again to Eq. (2.28) and proceeding similarly to the case with Dirichlet boundary conditions, we can express the modified cylinder kernel as

$$\tilde{T}(\mathbf{t}; \mathbf{q}, \mathbf{p}) = \frac{1}{2\pi\alpha\rho\rho' \sinh v} \left( \frac{\sinh(\frac{\pi}{2\alpha}v) \cos(\frac{\pi}{2\alpha}(\theta - \theta'))}{\cosh(\frac{\pi}{\alpha}v) - \cos(\frac{\pi}{\alpha}(\theta - \theta'))} - \frac{\sinh(\frac{\pi}{2\alpha}v) \cos(\frac{\pi}{2\alpha}(\theta + \theta'))}{\cosh(\frac{\pi}{\alpha}v) - \cos(\frac{\pi}{\alpha}(\theta + \theta'))} \right), \quad (5.20)$$

where  $v$  is defined as in Eq. (5.15). Using Eq.s (5.7-5.8) we obtain the renormalized VEV of the stress-energy tensor:

$$\langle 0 | \hat{T}_{\mu\nu}(\mathbf{q}) | 0 \rangle_{ren} \Big|_{\mu, \nu=0, \rho, \theta, z} =$$

$$A(\mathbf{q}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3\rho^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} - \left( \xi - \frac{1}{6} \right) \begin{pmatrix} -(B(\mathbf{q}) + C(\mathbf{q})) & 0 & 0 & 0 \\ 0 & B(\mathbf{q}) & -\rho E(\mathbf{q}) & 0 \\ 0 & -\rho E(\mathbf{q}) & \rho^2 C(\mathbf{q}) & 0 \\ 0 & 0 & 0 & B(\mathbf{q}) + C(\mathbf{q}) \end{pmatrix},$$

$$A(\mathbf{q}) := \frac{7\pi^4 + 8\alpha^4}{11520\pi^2\alpha^4\rho^4},$$

$$B(\mathbf{q}) := \frac{-3\pi^2 \cos(\frac{\pi\theta}{\alpha})(11\pi^2 - 2\alpha^2 + (\pi^2 + 2\alpha^2) \cos(\frac{2\pi\theta}{\alpha})) + 2\alpha^2(\pi^2 + 2\alpha^2) \sin^4(\frac{\pi\theta}{\alpha})}{96\pi^2\alpha^4 \sin^4(\frac{\pi\theta}{\alpha}) \rho^4},$$

$$C(\mathbf{q}) := \frac{6\pi^2 \cos(\frac{\pi\theta}{\alpha}) + (\pi^2 + 2\alpha^2) \sin^2(\frac{\pi\theta}{\alpha})}{16\pi^2\alpha^2 \sin^2(\frac{\pi\theta}{\alpha}) \rho^4}, \quad E(\mathbf{q}) := \frac{3\pi(3 + \cos(\frac{2\pi\theta}{\alpha}))}{32\alpha^3 \sin^3(\frac{\pi\theta}{\alpha}) \rho^4}. \quad (5.21)$$



Let us compare the above results with the ones of Eq. (5.17), holding for the case of Dirichlet boundary conditions on both the half-planes  $\pi_0, \pi_\alpha$ . As in Eq. (5.17), both the conformal and non-conformal parts of the renormalized stress-energy VEV diverge for  $\rho \rightarrow 0$ ; the latter also diverges for  $\theta \rightarrow 0, \alpha$ , so that Eq. (5.12) yields again a divergent pressure on the boundary. On the other hand, the present results differ from the ones derived in the previous subsection because of some crucial features; in particular, the conformal part has an overall minus sign and the function  $A(\mathbf{q})$  in Eq. (5.21) is always strictly positive (whereas the one in Eq. (5.17) changes sign for  $\alpha < \pi$  and  $\alpha > \pi$ ).

As for the boundary forces on  $\pi_\alpha$ , resorting to Eq. (5.11), in this case we obtain

$$p_i^{ren} = \frac{\delta_{i\theta}}{8\pi^2\rho^3} \left[ \frac{7\pi^4 + 8\alpha^4}{480\alpha^4} - \left( \xi - \frac{1}{6} \right) \frac{\pi^2 + 2\alpha^2}{\alpha^2} \right]. \quad (5.22)$$

We notice that also in this case the parameter  $\xi$  appears in the final expression for the renormalized pressure; because of this the resulting boundary forces can be either attractive or repulsive, depending on the value of  $\xi$ .

Again, we conclude comparing the results obtained for the renormalized stress-energy VEV and pressure for  $\alpha = \pi/2$  with the analogous ones deduced in subsection 4.10. Also this time, Eq.s (5.21) (5.22) (with  $\alpha = \pi/2$ ) are found to give, respectively, Eq.s (4.64) (4.65) with  $\alpha_1 = 1, \alpha_2 = -1$ .

**5.8 Neumann boundary conditions.** We now analyze the case where the field fulfills Neumann boundary conditions on both the half-planes  $\pi_0, \pi_\alpha$ . A complete orthonormal system of (improper) eigenfunctions  $(F_k)_{k \in \mathcal{K}}$  in  $L^2(\Omega)$  of the fundamental operator  $\mathcal{A}$ , with related eigenvalues  $(\omega_k^2)_{k \in \mathcal{K}}$ , is

$$F_k(\mathbf{q}) = \sqrt{\frac{\omega}{\pi\alpha}} J_{\lambda_n}(\omega\rho) \cos(\lambda_n\theta) e^{ihz}, \quad \lambda_n := \frac{n\pi}{\alpha}, \quad (5.23)$$

$$\omega_k^2 = \omega^2 + h^2 \quad \text{for } k \equiv (n, \omega, h) \in \mathcal{K} \equiv \mathbf{N}_0 \times (0, +\infty) \times \mathbf{R}$$

(recall that  $\mathbf{N}_0 := \{0, 1, 2, \dots\}$ ; again, we assume the measure on the label space  $\mathcal{K}$  is such that  $\int_{\mathcal{K}} dk \equiv \sum_{n=0}^{+\infty} \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} dh$ ). Also in this case, the modified cylinder kernel  $\tilde{T}$  can be evaluated according to Eq. (2.28). More precisely, proceeding as we did in subsection 5.6 for the case of Dirichlet boundary conditions (see, in particular, the derivation of Eq. (5.16)), we obtain

$$\tilde{T}(\mathbf{t}; \mathbf{q}, \mathbf{p}) = \quad (5.24)$$

$$\frac{1}{4\pi\alpha\rho\rho'\sinh v} \left( \frac{e^{\frac{\pi}{\alpha}v} - \cos(\frac{\pi}{\alpha}(\theta - \theta'))}{\cosh(\frac{\pi}{\alpha}v) - \cos(\frac{\pi}{\alpha}(\theta - \theta'))} + \frac{e^{\frac{\pi}{\alpha}v} - \cos(\frac{\pi}{\alpha}(\theta + \theta'))}{\cosh(\frac{\pi}{\alpha}v) - \cos(\frac{\pi}{\alpha}(\theta + \theta'))} \right);$$

again,  $v$  is defined as in Eq. (5.15). Now, we can resort once more to Eq.s (5.7-5.8) to evaluate the renormalized VEV of the stress-energy tensor; the result is

$$\begin{aligned} \langle 0 | \widehat{T}_{\mu\nu}(\mathbf{q}) | 0 \rangle_{ren} \Big|_{\mu, \nu=0, \rho, \theta, z} &= A(\mathbf{q}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3\rho^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \\ &+ \left( \xi - \frac{1}{6} \right) \left[ \begin{pmatrix} -(B(\mathbf{q})+C(\mathbf{q})) & 0 & 0 & 0 \\ 0 & B(\mathbf{q}) & -\rho E(\mathbf{q}) & 0 \\ 0 & -\rho E(\mathbf{q}) & \rho^2 C(\mathbf{q}) & 0 \\ 0 & 0 & 0 & B(\mathbf{q})+C(\mathbf{q}) \end{pmatrix} + G(\mathbf{q}) \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3\rho^2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \right], \end{aligned} \quad (5.25)$$

where the functions  $A, B, C, E$  are defined as in Eq. (5.17) and we set

$$G(\mathbf{q}) := \frac{\pi^2 - \alpha^2}{12\pi^2 \alpha^2 \rho^4}. \quad (5.26)$$

We notice that, in accordance with the existing literature (see, e.g., [11]), the conformal part of the renormalized stress-energy VEV coincides with the analogous contribution derived for Dirichlet boundary conditions in subsection 5.6; besides, comments analogous to those made at the end of the cited subsection also hold in this case. Let us only remark that the VEV  $\langle 0 | \widehat{T}_{\mu\nu}(\mathbf{q}) | 0 \rangle_{ren}$  has an additional term proportional to the function  $G(\mathbf{q})$ ; this function changes sign for either  $\alpha < \pi$  or  $\alpha > \pi$  and diverges for  $\rho \rightarrow 0$ .

Concerning the pressure on the boundary, also in this case Eq. (5.12) clearly yields a divergent result; on the other hand, using Eq. (5.11), we obtain

$$p_i^{ren} = -\frac{\delta_{i\theta}}{4\pi^2 \rho^3} \left[ \frac{\pi^4 - \alpha^4}{120\alpha^4} - \left( \xi - \frac{1}{6} \right) \frac{\pi^2 - \alpha^2}{\alpha^2} \right]. \quad (5.27)$$

As in the previous subsection, we find that the renormalized pressure depends of the parameter  $\xi$ .

Proceeding as explained in subsection 5.5, the renormalized stress-energy VEV (5.25) and pressure (5.27) are easily seen to give for  $\alpha = \pi$  and  $\alpha = \pi/2$ , respectively, Eq.s (4.62) (4.63) with  $\alpha_1 = 1$ , and Eq.s (4.64) (4.65) with  $\alpha_1 = \alpha_2 = 1$ .

**5.9 Periodic boundary conditions (the cosmic string).** Finally, let us consider the case where the field fulfills periodic boundary conditions on the half-planes  $\pi_0, \pi_\alpha$ , meaning that

$$\begin{aligned} \widehat{\phi}(t, \rho, 0, z) &= \widehat{\phi}(t, \rho, \alpha, z), \quad \partial_\theta \widehat{\phi}(t, \rho, 0, z) = \partial_\theta \widehat{\phi}(t, \rho, \alpha, z) \\ &\text{for } t, z \in \mathbf{R}, \rho \in (0, +\infty). \end{aligned} \quad (5.28)$$

In passing, let us mention that the same framework was also analysed by Dowker [13] and by Fulling et al. [23], both employing a point-splitting approach; more precisely, in [13] the conformal part of the energy density alone is computed, while in [23] the authors only report the graphs of the energy density and pressure for  $\xi = 1/4$ .

Similarly to the cases of the segment and parallel hyperplanes configurations with periodic boundary conditions (considered, respectively, in subsection 6.9 of Part I and in subsection 3.9 of the present paper), the spatial domain  $\Omega$  for the present setting is more properly addressed as a flat Riemannian manifold. The manifold  $\Omega$  has a global coordinate system  $\mathbf{q} = (\rho, \theta, z) : \Omega \rightarrow (0, +\infty) \times \mathbf{T}_\alpha^1 \times \mathbf{R}$ ,  $\mathbf{x} \mapsto \mathbf{q}(\mathbf{x})$  where the second factor is the one-dimensional torus  $\mathbf{T}_\alpha^1 := \mathbf{R}/(\alpha\mathbf{Z})$ ; the line element in these coordinates has the form (5.2) <sup>(15)</sup>. The corresponding spacetime  $\mathbf{R} \times \Omega$  (with the line element  $ds^2 = -dt^2 + d\ell^2$ ) is usually described in terms of a “cosmic string” due to the presence of a 1-dimensional topological defect coinciding with the axis  $\{\rho = 0\}$ .

A complete orthonormal system of (improper) eigenfunctions  $(F_k)_{k \in \mathcal{K}}$  of  $\mathcal{A}$  in  $L^2(\Omega)$ , with the related eigenvalues  $(\omega_k^2)_{k \in \mathcal{K}}$ , is given by

$$F_k(\mathbf{q}) := \sqrt{\frac{\omega}{2\pi\alpha}} J_{|\lambda_n|}(\omega\rho) e^{i\lambda_n\theta} e^{ihz}, \quad \lambda_n := \frac{2n\pi}{\alpha}, \quad (5.29)$$

$$\omega_k^2 := \omega^2 + h^2 \quad \text{for } k \equiv (n, \omega, h) \in \mathcal{K} \equiv \mathbf{Z} \times (0, +\infty) \times \mathbf{R}$$

(similarly to the previous subsections, we are assuming  $\mathcal{K}$  to be a measure space such that  $\int_{\mathcal{K}} dk \equiv \sum_{n=-\infty}^{+\infty} \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} dh$ ). The modified cylinder kernel  $\tilde{T}$  can then be evaluated starting from its eigenfunction expansion (2.28):

$$\begin{aligned} \tilde{T}(\mathbf{t}; \mathbf{q}, \mathbf{p}) &= \int_{\mathcal{K}} dk \frac{e^{-t\omega_k}}{\omega_k} F_k(\mathbf{x}) \overline{F_k}(\mathbf{y}) = \\ &= \frac{1}{2\pi\alpha} \sum_{n=-\infty}^{+\infty} e^{i\lambda_n(\theta-\theta')} \int_0^{+\infty} d\omega \omega J_{\lambda_n}(\omega\rho) J_{\lambda_n}(\omega\rho') \int_{-\infty}^{+\infty} dh \frac{e^{-t\sqrt{\omega^2+h^2}}}{\sqrt{\omega^2+h^2}} e^{ih(z-z')}. \end{aligned} \quad (5.30)$$

Evaluating the integrals in  $h$  and  $\omega$  as in [23] and considering separately the terms with positive and negative values of  $n$ , we obtain the expression

$$\tilde{T}(\mathbf{t}; \mathbf{q}, \mathbf{p}) = \frac{1}{2\pi\alpha\rho\rho' \sinh v} \left( 1 + \sum_{n=1}^{+\infty} e^{-\lambda_n v} e^{i\lambda_n(\theta-\theta')} + \sum_{n=1}^{+\infty} e^{-\lambda_n v} e^{-i\lambda_n(\theta-\theta')} \right), \quad (5.31)$$

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<sup>15</sup>In other words,  $\Omega$  is a quotient space of the Dowker manifold  $\Omega_\infty$ ; this is an infinite-sheeted Riemannian surface that can be described in terms of a global coordinate system

$$\mathbf{q} : \Omega_\infty \rightarrow (0, +\infty) \times \mathbf{R} \times \mathbf{R}, \quad \mathbf{x} \mapsto \mathbf{q}(\mathbf{x}) = (\rho(\mathbf{x}), \theta(\mathbf{x}), z(\mathbf{x}))$$

(and of the line element (5.2)).

which in turn, summing the geometric series, yields

$$\tilde{T}(\mathbf{t}; \mathbf{q}, \mathbf{p}) = \frac{1}{2\pi\alpha \rho \rho' \sinh v} \frac{\sinh(\frac{2\pi}{\alpha}v)}{\cosh(\frac{2\pi}{\alpha}v) - \cos(\frac{2\pi}{\alpha}(\theta - \theta'))} \quad (5.32)$$

(again,  $v$  is defined as in Eq. (5.15)). Using Eq.s (5.7-5.8) once more, we find the following expression for the renormalized stress-energy VEV:

$$\begin{aligned} & \langle 0 | \hat{T}_{\mu\nu}(\mathbf{q}) | 0 \rangle_{ren} \Big|_{\mu, \nu=0, \rho, \theta, z} = \\ & A(\mathbf{q}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3\rho^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \left( \xi - \frac{1}{6} \right) G(\mathbf{q}) \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3\rho^2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \\ & A(\mathbf{q}) = \frac{(2\pi)^4 - \alpha^4}{1440\pi^2\alpha^4\rho^4}, \quad G(\mathbf{q}) = \frac{(2\pi)^2 - \alpha^2}{24\pi^2\alpha^2\rho^4}. \end{aligned} \quad (5.33)$$

Let us observe that the above result does not depend explicitly on the angular variable  $\theta$ ; this was to be expected due to the homogeneity of the considered configuration with respect to this coordinate. Besides, as for the cases of Dirichlet and Neumann boundaries, both the conformal and non-conformal part of the renormalized VEV of the stress-energy tensor diverge near the axis  $\{\rho = 0\}$ , that is in the proximity of the cosmic string.

In conclusion, we notice that for  $\alpha = 2\pi$ , in which case the considered configuration is equivalent to that of a scalar field on the whole Minkowski spacetime, Eq. (5.33) gives  $A(\mathbf{q}) = G(\mathbf{q}) = 0$ . So, we have this result with its own interest: when zeta regularization is applied to a massless scalar field on the whole  $(3+1)$ -dimensional Minkowski spacetime, the renormalized stress-energy VEV vanishes identically.

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